Butterflies II: Torsors for 2-group stacks

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Abstract

We study torsors over 2-groups and their morphisms. In particular, we study the first non-abelian cohomology group with values in a 2-group. Butterfly diagrams encode morphisms of 2-groups and we employ them to examine the functorial behavior of non-abelian cohomology under change of coefficients. We re-interpret the first non-abelian cohomology with coefficients in a 2-group in terms of gerbes bound by a crossed module. Our main result is to provide a geometric version of the change of coefficients map by lifting a gerbe along the "fraction" (weak morphism) determined by a butterfly. As a practical byproduct, we show how butterflies can be used to obtain explicit maps at the cocycle level. In addition, we discuss various commutativity conditions on cohomology induced by various degrees of commutativity on the coefficient 2-groups, as well as specific features pertaining to group extensions.

Contents

1 Introduction

This paper is the second part of a series aimed at a systematic study of n-group stacks and their torsors. The first part, [AN09], is dedicated to the case $n = 2$ of 2-group stacks, or gr-stacks, in a slightly older terminology, and especially their morphisms. The most important result is that if 2-group stacks are made strict

by replacing them with (sheaves of) crossed modules, the groupoid of morphisms between 2-group stacks is equivalent to that of certain special diagrams called butterflies between corresponding crossed modules. This allows one to overcome the longstanding problem, even present in the non sheaf-theoretic setting, that replacing a monoidal category with a strict one is not a functorial construction.

Moving up one step in the cohomological ladder, the present paper, which is a direct sequel to [AN09], is concerned with the *torsors* for 2-group stacks. In a very general sense, torsors are the global geometric objects from which 1-cocycles with values in a 2-group stack arise, once suitable local trivializing data have been chosen. In effect, after a rigidification has been performed by replacing a 2-group stack by a crossed module, such cocycles will take values in a complex of sheaves (of length 2). This is the categorified version of the familiar process which associates to a principal G-bundle (or ordinary G-torsor) with local sections a 1-cocycle with values in G. Indeed the case $n = 1$ is the one of ordinary group objects. (In general a similar situation holds in the case of n-group stacks, as we shall see in later installments of this series.)

Our aim is to the study morphisms of torsors by harnessing the power of butterflies developed in the first part of this series, and to illustrate a few applications.

1.1 Content of the paper

It is useful to describe the context of our work in general terms. If $F\colon \mathscr{H} \to \mathscr{G}$ is a morphism of 2-group stacks over a certain site S, we want an appropriate morphism

$$F_*\colon \mathrm{TORS}(\mathscr{H}) \longrightarrow \mathrm{TORS}(\mathscr{G}), \tag{1.1.1}$$

where $\mathrm{TORS}(\mathscr{G})$ denotes the 2-stack of $\mathscr{G}$-torsors. One obtains in this way a geometric definition of degree-one non-abelian cohomology sets, with built-in functoriality. Namely, if by $\mathrm{TORS}(\mathscr{G})(*)$ we denote the 2-groupoid of global torsors, we can define $\mathrm{H}^1(\mathscr{G})$ simply as $\pi_0\big(\mathrm{TORS}(\mathscr{G})(*)\big)$, the connected components of that 2-groupoid; once F_* is defined, the functoriality of the first cohomology follows automatically.

A viable general mechanism by which torsors are extended "along" a morphism of n-group stacks is in fact well-known: given an $\mathscr{H}$-torsor $\mathscr{X}$, one defines F_* via the "contracted product"

$$F_*(\mathscr{X}) = \mathscr{X} \overset{\mathscr{H}}{\wedge} \mathscr{G}, \tag{1.1.2}$$

see [Bre90, §6], and section 6.1 below for all the details. The construction on the right-hand side above is the "categorification" of the standard one in the case of ordinary torsors, that is $n = 1$. The above definition of F_* provides a conceptual answer to finding a morphism (1.1.1), and therefore, by the above geometric definition of cohomology, an induced morphism

$$\mathrm{H}^1(\mathscr{H}) \longrightarrow \mathrm{H}^1(\mathscr{G}). \tag{1.1.3}$$

On the other hand, the recently introduced butterfly diagrams afford a rather fine-grained picture of morphisms of 2-group stacks, to be recalled below, so one asks for a similar description of (1.1.1) and the induced map (1.1.3).

To discuss this, let us recall from the first part that a butterfly allows us to decompose a morphism $F\colon \mathscr{H} \to \mathscr{G}$ into a "fraction"

$$\mathscr{H} \xleftarrow{\ Q\ } \mathscr{E} \xrightarrow{\ P\ } \mathscr{G}\,,$$

where Q is an equivalence of 2-group stacks. Actually, if we introduce crossed modules $H_1 \to H_0$ and $G_1 \to G_0$ for $\mathscr{H}$ and $\mathscr{G}$, respectively, the fraction above is determined by a **butterfly diagram** of group objects:

$$\begin{tikzcd}
H^{-1} \arrow[dd] \arrow[dr] & & G^{-1} \arrow[dl] \arrow[dd] \\
& E \arrow[dl] \arrow[dr] & \\
H^0 & & G^0
\end{tikzcd}$$

The NW-SE sequence is a complex, and the NE-SW sequence is a group extension. One finds the resulting map $H_1 \times G_1 \to E$ is a crossed module in its own right, which is quasi-isomorphic to $H_1 \to H_0$, and determines the stack $\mathscr{E}$. In sum, with a butterfly we can split $F\colon \mathscr{H} \to \mathscr{G}$ into a fraction of morphisms corresponding to morphisms of crossed modules. In fact the butterfly corresponds to a fraction in the derived category of crossed modules

$$H_\bullet \xleftarrow{\ q\ } E_\bullet \xrightarrow{\ p\ } G_\bullet\,, \tag{1.1.4}$$

where now p and q are genuine morphisms of crossed modules (the latter being a quasi-isomorphism) inducing the corresponding ones denoted by upper-case letters between corresponding 2-group stacks.

We have alluded to the fact that classes in, say, $\mathrm{H}^1(\mathscr{G})$ can be represented by 1-cocycles with values in the crossed-module $G_1 \to G_0$. Let us remind the reader, following [Bre90], that such cocycles can equivalently be described as simplicial maps from hypercovers of objects of S to a reasonable model of the classifying space of $\mathscr{G}$. One such is provided, for instance, by the $\overline{\mathrm{W}}$ construction applied to the simplicial group $\underline{G}$ determined by the crossed module. It is possible to prove, using (1.1.2), that a cocycle with values in $H_\bullet$ determines one with values in $G_\bullet$. The argument mostly rests on the construction of a morphism

$$\overline{\mathrm{W}}\,\underline{H} \longrightarrow \overline{\mathrm{W}}\,\underline{G}$$

between classifying objects. (Note, in passing, that this is the very definition of *weak* morphism of crossed modules in the set-theoretic case, see [Noo05].) Unfortunately, starting from the morphism F as a whole is not very explicit or constructive, not only because it requires a chosen rigidification of .the otherwise

weak group laws of $\mathscr{H}$ and $\mathscr{G}$, but chiefly because F does not determine a direct morphism $H_\bullet \to G_\bullet$ between crossed modules.

Our first result is to exploit the butterfly technology to provide a much more direct approach to computing the morphism (1.1.3). As explained in section 4 below, the morphism (1.1.3) can be computed by, figuratively speaking, lifting a 1-cocycle, or equivalently a simplicial map $\eta\colon U_\bullet \to \overline{\mathrm{W}}\,\underline{H}$ along the diagram (1.1.4). More concretely, one constructs a new simplicial map $\eta'\colon U_\bullet \to \overline{\mathrm{W}}\,\underline{E}$ such that its projection via q is η (possibly after passing to a finer hypercover which will not be notationally distinguished); in effect η' represents the same class as η, since $H_\bullet$ and $E_\bullet$ are quasi-isomorphic. Then the sought-after morphism is simply obtained by projecting η' along p. Diagrammatically, we have:

$$\eta \xleftarrow{\;q\;} \eta' \xrightarrow{\;p\;} \xi$$

where ξ denotes the resulting simplicial map or 1-cocycle with values in $G_\bullet$.

This same method, in simpler form, works for 0-cocycles as well, such as those dealt with in the first part, and it is expected to do so for higher degree classes in the case the 2-group stacks involved are symmetric or Picard.

The construction just outlined embodies the general idea that informs our main result, a novel geometric construction of the morphism (1.1.1). Starting from the butterfly decomposition of F we want to decompose F_* as

$$\mathrm{TORS}(\mathscr{H}) \xleftarrow{\;Q_*\;} \mathrm{TORS}(\mathscr{E}) \xrightarrow{\;P_*\;} \mathrm{TORS}(\mathscr{G}),$$

where P_* and Q_* are expected to be simpler than F_*, since P and Q each arise from a strict morphism. Moreover, this decomposition should be such that passing to cohomology classes, or better yet to representative cocycles, provides a calculation of the map (1.1.3) of cohomology sets outlined above.

Now, in practice, we do not implement our program within the context of torsors over a 2-group stack, essentially due to the fact that the direction of P is at odds with the natural notion of extension of torsors along a morphism (i.e. P goes in the wrong direction). One can of course make the choice of a quasi-inverse P^* to it, but that defeats the purpose, so to speak; we want something more canonical.

It turns out the concept of gerbes "bound" by a crossed module is the appropriate notion. In very broad terms, the general idea, originally due to Debremaeker (see [Deb77]), is that a gerbe $\mathscr{P}$ bound by a crossed module $G_1 \to G_0$ is a gerbe equipped with a morphism

$$\mu\colon \mathscr{P} \longrightarrow \mathrm{TORS}(G_0)$$

subject to certain additional conditions, recalled in section 5, which in particular make $\mathscr{P}$ into a G_1-gerbe. These gerbes give rise to non-abelian cohomology classes with values in the crossed module (or in fact in a 2-group stack) too. Torsors do the same of course, and indeed we prove there in an equivalence

$$\mathrm{TORS}(\mathscr{G}) \longrightarrow \mathrm{GERBES}(G_1, G_0), \tag{1.1.5}$$

which generalizes a similar result of Breen (for the 2-group stack of G-bitorsors for a group object G and G-gerbes) put forward in [Bre90]. While the equivalence and the statement have pretty much identical forms, the proof is however quite different, and we have included it here.

Thus the actual version of the decomposition we provide is is to *define* a morphism

$$F_+ \colon \operatorname{GERBES}(H_1, H_0) \longrightarrow \operatorname{GERBES}(G_1, G_0)$$

by means of the following diagram

$$\operatorname{GERBES}(H_1, H_0) \xleftarrow{Q_+^0} \operatorname{GERBES}(E_1, E_0) \xrightarrow{P_+^0} \operatorname{GERBES}(G_0, G_1)$$

where the definition of P_+^0 and Q_+^0 is direct (available in [Deb77]), since p and q are *strict* morphisms of crossed modules. The quasi-inverse to the arrow pointing to the left, is surprisingly simple in the gerbe context: from a gerbe $\mathscr{Q}$ bound by the crossed module $H_\bullet$, the gerbe bound by $E_\bullet$ that we need is simply the stack fibered product:

$$\mathscr{Q}' = \mathscr{Q} \times_{\operatorname{TORS}(H_0)} \operatorname{TORS}(E).$$

The image of $\mathscr{Q}'$ by Q_+ is equivalent to $\mathscr{Q}$, and by "pushing" along P, that is, considering the image under P_+, we obtain a gerbe bound by $G_\bullet$. We then prove, essentially by comparing cohomology classes, that F_+ is equivalent to F_*, modulo the equivalence (1.1.5), so in other words we obtain a square

$$\begin{tikzcd}
\operatorname{TORS}(\mathscr{H}) \arrow[r, "F_*"] \arrow[d] & \operatorname{TORS}(\mathscr{G}) \arrow[d] \\
\operatorname{GERBES}(H_1, H_0) \arrow[r, "F_+"] & \operatorname{GERBES}(G_1, G_0)
\end{tikzcd}$$

commuting up to natural isomorphism.

After having gone through these general results, we move on to consider some applications, mainly to the abelian structures on cohomology resulting when braided, symmetric, or Picard structures are imposed on the coefficients, and specifically when group extensions in the sense of Grothendieck ([Gro72]) and Breen ([Bre90, §8]) are concerned. In the end we make contact with the definition of weak morphism between crossed modules as simplicial maps between classifying spaces. Since several results are already known, our discussion assumes a more informal character compared to the previous sections, and many arguments are just sketched.

Let us conclude with a comment about the use of gerbes bound by crossed modules. The original intent behind the introduction of the concept of gerbe bound by a crossed module was to correct the perceived lack of functoriality inherent in Giraud's definition of higher non-abelian cohomology using liens (see [Gir71]). Functoriality was addressed in Debremaeker's paper [Deb77] by considering only morphisms of crossed modules, that is what we now call strict

morphisms. This restriction to strict morphisms is not the natural thing to do, and since non abelian cohomology depends on the associated 2-group stack, rather than on the coefficient crossed module itself, introducing torsors led to a better conceptual understanding of the functoriality of non abelian cohomology. Thus the notion has not been developed or used until recently, when it became useful in different contexts (see for instance [Ald08; Mil03]).

This state of affairs has been changed by the better control of morphisms afforded by the use of butterflies, since they allow a description of all morphisms by way of crossed modules. Thus now the use of gerbes bound by crossed modules *plus* the use of butterflies affords a geometrization of the non abelian derived category equivalent to the one obtained by using the torsor picture.

1.2 Organization of the paper

Here is a brief synopsis of this paper's content. Since this is a direct continuation of [AN09], the reader will unavoidably be constantly referred to that paper. In order to make this process a little less burdensome, we recall in section 2 some of the results of the first part that we shall most often need here. In section 3 we have collected results and definitions concerning torsors over gr-stacks and non-abelian cohomology. Our purpose was of course to make a moderate attempt at being self-contained and at a uniformity of conventions.

By design the material in these sections is not new, except maybe in the presentation. New results begin in earnest in section 4, where we explicitly describe in terms of butterflies the morphism of non-abelian first cohomology sets induced by a morphism of gr-stacks.

In section 5 we present the idea of a gerbe bound by a crossed module, originally due to Debremaeker. In addition to re-introduce the main definitions, we analyze the local structure and prove the cohomology class determined by such an object takes values in the gr-stack associated to the crossed module. Since this is almost the same idea as that of a torsor for said gr-stack, we determine the precise relation between the two. In this way we obtain a generalization of an analogous result due to [Bre90, Proposition 7.3]. The sort of rigidification that the passage from $G \to \mathrm{Aut}(G)$ to a general crossed module $G \to \Pi$ entails makes the proof very different, so we discuss it in detail.

The morphism of first non-abelian cohomology sets induced by a morphism of gr-stacks discussed in purely algebraic terms in section 4 has a well-known geometric realization in terms of extension of torsors along that morphism (this is the categorification of the well-known extension of structural groups for principal bundles). The analogous procedure in terms of gerbes bound by crossed modules is described in section 6. It generalizes Debremaeker's notion of morphism of gerbes bound by crossed modules, which only uses what we call *strict* morphisms of crossed modules. The general case is treated in section 6.3. We prove that the morphism so obtained is equivalent, modulo the equivalence between torsors and gerbes, to the morphism given by the extension of torsors, and in section 6.4 we show that the induced cohomology class is precisely the one computed by the procedure described in section 4.

Sections 7 and 8 are devoted to some applications. In section 7 we briefly analyze the commutativity conditions on cohomology ensuing from the assumption that the coefficient crossed module (or gr-stack) be at least braided. It is well-known that in this case the first a priori non-abelian cohomology acquires a group structure which becomes abelian if the coefficient gr-stack is symmetric. Our approach is to analyze these structures in terms of specific butterfly diagrams associated to braided crossed modules which express the fact that for a braided gr-stack the monoidal structure is a weak morphism. This is discussed in detail in [AN09, §7]. Using these special butterflies, we are in position to apply the general theory of section 6 to obtain a novel description of the group structures on cohomology, for which we can write explicit product formulas at the cocycle level. Section 8 contains some remarks about group extensions. First about how the classical Schreier theory of extensions, from the geometric perspective of Grothendieck and Breen, fits in the butterfly framework. We then discuss again commutative structures, and to some extent abelianization maps. Some final informal paragraphs are devoted to making contact with the simplicial definition of weak morphism of crossed modules.

1.3 Conventions and notations

In the sequel we shall refer to [AN09] simply as "Part I." We keep its standing assumptions, notations, and typographical conventions: in particular, S denotes quite generally a site with subcanonical topology, and $\mathsf{T} = \mathsf{S}^\sim$ denotes the topos of Set-valued sheaves over S. Again as in Part I we break our convention usage in the introduction by reverting to the older term "gr-stack" in place of the more recent 2-group (stack). Concerning the numbering scheme, references to the first part are made using that paper's numbering sequence. For this one, we have chosen to cut the numbering off by one level, due to its reduced length (compared to [Part I]).

2 Recollection of results from [Part I]

2.1 Crossed modules and gr-stacks

Let $\mathscr{G}$ be a gr-stack (or 2-group stack), that is a stack over S endowed with a group-like monoidal structure

$$\otimes \colon \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G},$$

see, for example, [Bre90; Bre92; Bre94a], and [Sí75; JS93] for the point-wise case. Many of the results from the previous references which are required in this text are summarized in [Part I], to which the reader is referred for more details. Here we limit ourselves to recall that starting from $\mathscr{G}$ we can always construct a homotopy fibration

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{\pi_{\mathscr{G}}} \mathscr{G},$$

where $\partial\colon G_1 \to G_0$ has the structure of a crossed module, so that in fact $\mathscr{G}$ can be recovered as its associated gr-stack. More precisely, the crossed module $G_1 \to G_0$ provides us with a concrete model for the associated gr-stack, namely there is an equivalence

$$\mathscr{G} \xrightarrow{\sim} \mathrm{TORS}(G_1, G_0).$$

Following Deligne [Del79], the right-hand side denotes the stack of those G_1-torsors which become trivial after extension $P \rightsquigarrow P \wedge^{G_1} G_0$. Thus, $\mathscr{G}$ is realized as the homotopy fiber

$$\mathscr{G} \longrightarrow \mathrm{TORS}(G_1) \xrightarrow{\partial_*} \mathrm{TORS}(G_0),$$

where an object of $\mathscr{G}$ is a pair (P, s), comprising a right G_1-torsor P and a trivialization $s\colon P \wedge^{G_1} G_0 \xrightarrow{\sim} G_0$. When combined with the crossed module structure, this picture allows us to realize $\mathscr{G}$ as a sub-gr-stack of $\mathrm{BITORS}(G_1)$ by observing that the underlying G_1-torsor in the pair (P, s) acquires a G_1-bitorsor structure by defining a left G_1-action through s as:

$$g \cdot p := p\, g^{s(p)},$$

where $p \in P$, $g \in G_1$, and s is viewed as a G_1-equivariant morphism $s\colon P \to G_0$. A morphism $\varphi\colon (P, s) \to (Q, t)$ in $\mathscr{G}$ is therefore a commutative diagram

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr, bend right, "s"'] & & Q \arrow[dl, bend left, "t"] \\ & G_0 & \end{tikzcd} .$$

It follows that the monoidal structure of $\mathscr{G}$ can be expressed through standard contraction of bitorsors: for two objects (P, s) and (Q, t) of $\mathscr{G}$ we set

$$(P, s) \otimes (Q, t) = (P \overset{G_1}{\wedge} Q, s \wedge t),$$

where $s \wedge t$ is the G_1-equivariant map given by $(p, q) \mapsto s(p)t(q)$, where (p, q) represents a point of $P \wedge^{G_1} Q$. It results from the compound trivialization:

$$\big(P \overset{G_1}{\wedge} Q\big) \overset{G_1}{\wedge} G_0 \simeq P \overset{G_1}{\wedge} \big(Q \overset{G_1}{\wedge} G_0\big) \xrightarrow{1 \wedge t} P \overset{G_1}{\wedge} G_0 \xrightarrow{s} G_0.$$

In dealing with gr-stacks and crossed modules we will always—often tacitly—make use of the interplay outlined in the previous paragraphs, and therefore move freely between gr-stacks and crossed modules.

2.2 Butterflies and weak morphisms

Let $H_\bullet$ and $G_\bullet$ be crossed modules of T, and let $\mathscr{H}$ and $\mathscr{G}$ denote their associated gr-stacks, respectively.

A morphism $F\colon \mathscr{H} \to \mathscr{G}$, that is, an additive functor, is by definition a *weak morphism* from $H_\bullet$ to $G_\bullet$. All weak morphisms from $H_\bullet$ to $G_\bullet$ form a groupoid, denoted $\mathsf{WM}(H_\bullet, G_\bullet)$.

A *butterfly* from $H_\bullet$ to $G_\bullet$ is by definition a commutative diagram of group objects of T:

(2.2.1)
$$
\begin{tikzcd}
H_1 \arrow[dd, "\partial"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\
& E \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\
H_0 & & G_0
\end{tikzcd}
$$

such that the NW-SE sequence is a complex, and the NE-SW sequence is a group extension. The various maps satisfy the equivariance conditions written set-theoretically as:

(2.2.2)
$$
\imath(g^{\jmath(e)}) = e^{-1}\imath(g)e, \quad \kappa(h^{\pi(e)}) = e^{-1}\kappa(h)e
$$

where $g \in G_1, h \in H_1, e \in E$. An easy consequence of (2.2.2) is that the images of $\jmath$ and κ commute in E.

The short-hand notation $[H_\bullet, E, G_\bullet]$ will be used for a butterfly from $H_\bullet$ to $G_\bullet$.

A *morphism of butterflies* $\varphi\colon [H_\bullet, E, G_\bullet] \to [H_\bullet, E', G_\bullet]$ is given by a group isomorphism $\varphi\colon E \xrightarrow{\sim} E'$ such that the diagram:

$$
\begin{tikzcd}
H_1 \arrow[r] \arrow[dr] \arrow[dd] & E' \arrow[ddl, bend right] \arrow[ddr, bend left] & G_1 \arrow[l] \arrow[dl] \arrow[dd] \\
& E \arrow[u] \arrow[dl] \arrow[dr] & \\
H_0 & & G_0
\end{tikzcd}
$$

commutes and is compatible with all the conditions involved in diagram (2.2.1). Two morphisms are composed in the obvious way. In this way butterflies from $H_\bullet$ to $G_\bullet$ form a groupoid, denoted $\mathsf{B}(H_\bullet, G_\bullet)$.

One of the main results of [Part I, Theorem 4.3.1] reads, in part:

2.2.3 Theorem. *There is an equivalence of groupoids*

$$
\mathsf{B}(H_\bullet, G_\bullet) \xrightarrow{\sim} \mathsf{WM}(H_\bullet, G_\bullet).
$$

A pair of quasi-inverse functors

$$
\Phi\colon \mathsf{B}(H_\bullet, G_\bullet) \longrightarrow \mathsf{WM}(H_\bullet, G_\bullet)
$$

and

$$
\Psi\colon \mathsf{WM}(H_\bullet, G_\bullet) \longrightarrow \mathsf{B}(H_\bullet, G_\bullet).
$$

is explicitly described in Part I.

Strict morphisms of crossed modules (described in detail in Part I, section 3.2) correspond to butterfly diagrams whose NE-SW diagonal is split—with a definite choice of the splitting morphism, see Part I, section 4.5. Conversely, a *splittable* butterfly, namely one whose NE-SW diagonal is in the same isomorphism class as a semi-direct product, by definition corresponds to a morphism equivalent to a strict one.

A butterfly diagram is called *flippable* or *reversible* if both diagonal are extensions. The corresponding weak morphism is an equivalence.

It easy to verify that from the butterfly diagram (2.2.1) the homomorphism

$$\partial_E \colon H_1 \times G_1 \longrightarrow E,$$

where $\partial_E(h, g) = \kappa(h)\imath(g)$, is a crossed module with the obvious action of E on $H_1 \times G_1$ through that of H_0 and G_0 on the respective factors. Let us denote this crossed module by

$$E_\bullet \colon E_1 \to E_0,$$

with $E_0 = E$ and $E_1 = H_1 \times G_1$.

From Part I we have that the weak morphism given by the butterfly (2.2.2) factorizes as a "fraction"

$$H_\bullet \xleftarrow{\sim} E_\bullet \longrightarrow G_\bullet$$

of strict morphisms of crossed modules. The one to the left is a quasi-isomorphisms, that is, it induces isomorphisms on the corresponding homotopy sheaves:

$$\pi_i(E_\bullet) \simeq \pi_i(H_\bullet), \quad i = 0, 1.$$

2.3 Composition of butterflies and the bicategory of crossed modules

Composition of butterflies is by juxtaposition: Given two butterflies

$$\begin{tikzcd}
K_1 \arrow[dd, "\partial_K"'] \arrow[dr] & & H_1 \arrow[dl, "\imath'"'] \arrow[dd, "\partial_H"] \\
& F \arrow[dl] \arrow[dr, "\jmath'"] & \\
K_0 & & H_0
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1 \arrow[dd, "\partial_H"'] \arrow[dr, "\kappa"] & & G_1 \arrow[dl] \arrow[dd, "\partial_G"] \\
& E \arrow[dl, "\pi"'] \arrow[dr] & \\
H_0 & & G_0
\end{tikzcd}$$

their composition is the butterfly (defined set-theoretically in [Noo05]):

$$\begin{tikzcd}
K_1 \arrow[dd, "\partial_K"'] \arrow[dr] & & G_1 \arrow[dl] \arrow[dd, "\partial_G"] \\
& F \times^{H_1}_{H_0} E \arrow[dl] \arrow[dr] & \\
K_0 & & G_0
\end{tikzcd}$$

The center is given by a kind of pull-back/push-out construction: we take the fiber product $F \times_{H_0} E$ and mod out the image of H_1 (see also [Part I, §5.1], for details).

This composition is *not* associative: if $[L_\bullet, M, K_\bullet]$ is a third butterfly, then the construction of the composite only yields an *isomorphism*

$$\big(M \times^{K_1}_{K_0} F\big) \times^{H_1}_{H_0} E \xrightarrow{\sim} M \times^{K_1}_{K_0} \big(F \times^{H_1}_{H_0} E\big).$$

An almost immediate consequence is

2.3.1 Theorem (Part I, Theorem 5.1.4)**.** *When equipped with the morphism groupoids* $\mathsf{B}(-,-)$*, crossed modules in* T *form a bicategory, denoted* $\underline{\mathsf{XMod}}(\mathsf{S})$*.*

There are fibered analogs of the various entities we have introduced so far: so, for instance, one defines a fibered category $\mathscr{B}(H_\bullet, G_\bullet)$, which is defined as usual by assigning to $U \in \operatorname{Ob}\mathsf{S}$ the groupoid

$$\mathsf{B}(H_\bullet|_U, G_\bullet|_U),$$

and to every arrow $V \to U$ of S the functor

$$\mathsf{B}(H_\bullet|_U, G_\bullet|_U) \longrightarrow \mathsf{B}(H_\bullet|_V, G_\bullet|_V).$$

Starting from $\mathsf{WM}(H_\bullet, G_\bullet)$ instead, an identical procedure leads to a fibered category $\mathscr{WM}(H_\bullet, G_\bullet)$ over S. It is proved in [Part I, 4.6.1, 4.6.2] that both are stacks (in groupoids) over S. In a more general, but similar, fashion, the bicategory $\underline{\mathsf{XMod}}(\mathsf{S})$ has a fibered analog, denoted $\mathfrak{XMod}(\mathsf{S})$. Thanks to the fact that $\mathscr{B}(H_\bullet, G_\bullet)$ is itself a stack, $\mathfrak{XMod}(\mathsf{S})$ is a pre-bistack over S. On the other hand, gr-stacks form a 2-stack denoted GR-STACKS(S), hence the obvious morphism $\mathfrak{XMod}(\mathsf{S}) \to$ GR-STACKS(S) sending a crossed module to its associated gr-stack is 2-faithful. Moreover, every gr-stack $\mathscr{G}$ is equivalent to the gr-stack associated to a crossed module—see [Part I, Proposition 5.3.7]. Therefore the above morphism is essentially surjective, and it follows that $\mathfrak{XMod}(\mathsf{S})$ is a bistack.

3 Torsors and non-abelian cohomology

In this section we recall some facts about $\mathscr{G}$-torsors, where $\mathscr{G}$ is a gr-stack. This is necessary in order to compare them with one of the main objects of study in this text, the gerbes bound by the crossed module $G_1 \to G_0$ whose associated gr-stack is $\mathscr{G}$. Those gerbes will be introduced in section 5. Since we shall also be concerned with classes of equivalence of such objects, as well as functoriality properties, it is useful to go through a quick review of some definitions in non-abelian cohomology.

3.1 Non-abelian cohomology

Let us recall the main definitions, following [Bre90] and [Ill71; Jar89; Jar86]. Let $\underline{G}$ be a simplicial group-object of T. The non-abelian cohomology with values in $\underline{G}$ can be defined as

$$\mathrm{H}^i(*, \underline{G}) = \begin{cases} \mathrm{Hom}_{\mathcal{D}(\mathsf{T})}(*, \Omega^{-i}\,\underline{G}), & i \leq 0, \\ \mathrm{Hom}_{\mathcal{D}(\mathsf{T})}(*, \mathrm{B}\,\underline{G}), & i = 1. \end{cases}$$

Here $*$ denotes the terminal object of T, Ω denotes the loop construction, whereas $\mathrm{B}\,\underline{G}$ is some (in fact any) form for the classifying space construction, for example $\overline{\mathrm{W}}\,\underline{G}$. $\mathcal{D}(\mathsf{T})$ denotes the derived category of simplicial objects of T in the same sense as [Ill71; Bre90], that is, by localizing at the morphisms of simplicial objects that induce isomorphisms of homotopy sheaves.

Note that the simplicial group structure is only relevant in order to define H^1, whereas for all other degrees $i \leq 0$ the definition only uses the underlying simplicial set structure. But also note that the former will only be a pointed set, as opposed to the others which carry group structures (abelian for $i < 0$). If we use the convention that $\mathrm{B}^{-1} \stackrel{\text{def}}{=} \Omega$, the various $\mathrm{H}^i(*, \underline{G})$ are computed as a colimit:

$$\mathrm{H}^i(*, \underline{G}) = \varinjlim_{V \to *} \big[*, \mathrm{B}^i\,\underline{G} \big],$$

where the colimit runs over homotopy classes of hypercovers of $*$ and $[-,-]$ denotes (simplicial) homotopy classes.

Our main focus will be the pointed set $\mathrm{H}^1(*, \underline{G})$ when the coefficient simplicial group arises from a crossed module $G_1 \to G_0$, which we denote by $\mathrm{H}^1(*, G_1 \to G_0)$. In view of the fact that any gr-stack $\mathscr{G}$ can be realized as the gr-stack associated to a crossed module $G_1 \to G_0$, as explained in Part I, we can write the same cohomologies by emphasizing the stack, rather than the crossed module, as coefficients, as $\mathrm{H}^i(*, \mathscr{G})$, $i \leq 1$. In fact more stress will be put on the *cocycles* representing cohomology classes, rather than on the classes themselves. After all, the former naturally arise from any appropriate decomposition (i.e. local description) of geometric objects, such as torsors and gerbes, as it will be clear below.

Following [Bre90], it will be convenient to recall the simplicial definition of 1-cocycles, as well as the more geometric one that simply categorifies the standard definition by replacing a group with a gr-stack.

3.2 1-Cocycles with values in crossed modules

If $\underline{G}_\bullet$ is a simplicial group object of T, there is a model for its classifying space provided by the $\overline{\mathrm{W}}$-construction. Namely, $\overline{\mathrm{W}}\,\underline{G}_\bullet$ is the simplicial object of T given by:

$$\overline{\mathrm{W}}\,\underline{G}_0 = *, \qquad \overline{\mathrm{W}}\,\underline{G}_n = \underline{G}_0 \times \underline{G}_1 \times \cdots \times \underline{G}_{n-1}\,, \quad n \geq 1.$$

The face and degeneracy maps are:

$$d_i(\underline{g}_0,\dots,\underline{g}_{n-1}) = \begin{cases} (d_1\underline{g}_1,\dots,d_{n-1}\underline{g}_{n-1}) & i=0 \\ (\underline{g}_0,\dots,\underline{g}_{i-1}d_0\underline{g}_i,\underline{g}_{i+1},\dots,d_{n-i-1}\underline{g}_{n-1}) & 0<i<n \\ (\underline{g}_0,\dots,\underline{g}_{n-2}) & i=n \end{cases}$$

and

$$s_i(\underline{g}_0,\dots,\underline{g}_{n-1}) = \begin{cases} (\underline{1},s_0\underline{g}_0,\dots,s_{n-1}\underline{g}_{n-1}) & i=0 \\ (\underline{g}_0,\dots,\underline{g}_{i-1},\underline{1},s_0\underline{g}_i,\dots,s_{n-i-1}\underline{g}_{n-1}) & 0<i<n \\ (\underline{g}_0,\dots,\underline{g}_{n-1},\underline{1}) & i=n \end{cases}$$

We have slightly changed the formulas of ref. [May92, §21] in order to better fit with our "action on the right" convention.

If G is a group object of T, identified with the constant simplicial group, then the previous construction reduces to the standard classifying simplicial space $\mathrm{B}\,G$.

3.2.1 Definition. Let $V_\bullet \to U$ be a hypercover. A 1-*cocycle over* U is a simplicial map $\xi\colon V_\bullet \to \overline{\mathrm{W}}\,\underline{G}_\bullet$. Two such cocycles ξ,ξ' are *equivalent* if there is a simplicial homotopy $\alpha\colon \xi \Rightarrow \xi'\colon V_\bullet \to \overline{\mathrm{W}}\,\underline{G}_\bullet$.

Let $\underline{G}_\bullet$ be the nerve of the groupoid G determined by a crossed module $G_1 \to G_0$. In this case we have $\overline{\mathrm{W}}\,\underline{G}_1 = G_0, \overline{\mathrm{W}}\,\underline{G}_2 = G_0 \times (G_0 \times G_1), \overline{\mathrm{W}}\,\underline{G}_3 = G_0 \times (G_0 \times G_1) \times (G_0 \times G_1 \times G_1)$, etc. A simplicial map $\xi\colon V_\bullet \to \overline{\mathrm{W}}\,\underline{G}_\bullet$ will be determined by its 3-truncation ([Bre90]).

A rather tedious, but otherwise straightforward calculation shows that the simplicial map ξ determines, and is determined by, a pair (x,g) where $x\colon V_1 \to G_0$ and $g\colon V_2 \to G_1$ satisfying the condition

$$d_1^*x = d_2^*x\, d_0^*x\,\partial g \tag{3.2.2a}$$

$$d_0^*g\, d_2^*g = (d_3^*g)^{(d_0d_1)^*x}\, d_1^*g \tag{3.2.2b}$$

and the normalizations $s_0^*x = 1$, $s_0^*g = s_1^*g = 1$. The explicit expressions of the maps ξ_i, $i = 0,\dots,3$ are as follows: $\xi_0 = *, \xi_1 = x\colon V_1 \to G_0$, whereas $\xi_2\colon V_2 \to G_0 \times (G_0 \times G_1)$ and $\xi_3\colon V_3 \to G_0 \times (G_0 \times G_1) \times (G_0 \times G_1 \times G_1)$ are given by

$$\xi_2 = (d_2^*x,(d_0^*x,g))$$
$$\xi_3 = ((d_2d_3)^*x,((d_0d_3)^*x,d_3^*g),((d_0d_1)^*x,d_0^*g,(d_0^*g)^{-1}d_1^*g).$$

3.2.3 Remark. There exists a compelling way of organizing the above data. The idea is that from the form of ξ_1 and ξ_2 we can use x as a label for a 1-cell of $\overline{\mathrm{W}}\,\underline{G}_\bullet$, and g as a label for the 2-cells. With this in mind, equation (3.2.2a)

represents a 2-cell with its boundary, as in the following diagram:

$$
\begin{tikzcd}
0 \arrow[rr, "d_2^* x"] \arrow[dr, bend right, "d_1^* x"'] & \arrow[d, phantom, "g"] & 1 \arrow[dl, bend left, "d_0^* x"] \\
& 2 &
\end{tikzcd}
$$

Similarly, (3.2.2b) represents the compatibility of the four possible pullbacks of (3.2.2a), and therefore has a tetrahedral shape:

$$
\begin{tikzcd}
& 3 & \\
0 \arrow[ur, "(d_1 d_2)^* x"] \arrow[rr, "(d_1 d_3)^* x"'] \arrow[dr, "(d_2 d_3)^* x"'] & & 2 \arrow[ul, "(d_0 d_1)^* x"'] \\
& 1 \arrow[uu, "(d_0 d_2)^* x"] \arrow[ur, "(d_0 d_3)^* x"'] &
\end{tikzcd}
$$

We have not recorded the face labels to avoid cluttering the diagram. To recover them, and hence equation (3.2.2b), observe that for $i \in \mathbf{3} = \{0, \dots, 3\}$, $d_i^* g$ is the 2-cell with vertices given by the complement of i in $\mathbf{3}$.

Alternatively, the following planar version is perhaps clearer:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr, phantom, "d_2^* g"' near start] & 3 \\
1 \arrow[r] \arrow[ur] \arrow[ur, phantom, "d_0^* g"' near end] & 2 \arrow[u]
\end{tikzcd}
=
\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "d_1^* g" near start, shift left=3ex] & 3 \\
1 \arrow[r] \arrow[ur, phantom, "d_3^* g" near start, shift right=3ex] & 2 \arrow[u]
\end{tikzcd}
$$

Note also that the 2-cell $d_3^* g$ is the only one *not* including the vertex 3. Hence an action by $(d_0 d_1)^* x$ is required. Also, the right action should match composition, so that the 2-cells should be traversed from bottom to top, relative to the last diagram.

Some aspects of the above constructions, in particular the seemingly arbitrary labeling of the vertices, given that $\overline{\mathrm{W}}\, \underline{G_\bullet}$ has only one 0-cell, may appear somewhat arbitrary. A full geometric explanation will be possible after the connection with trivializations of torsors and (equivalently) gerbes is made in sections 3.4 and 5.4, respectively.

A **simplicial homotopy** $\alpha \colon \xi \to \xi'$ is uniquely determined by $y \colon V_0 \to G_0$ and $a_0, a_1 \colon V_1 \to G_1$ such that:

$$
\begin{aligned}
(d_1^* y)\, x' &= x\, (d_0^* y)\, \partial(a_1 a_0^{-1}) \\
d_0^*(a_1 a_0^{-1})\, d_2^*(a_1 a_0^{-1})^{d_0^* x'} g' &= g^{(d_0 d_1)^* y}\, d_1^*(a_1 a_0^{-1})
\end{aligned}
\tag{3.2.4}
$$

Note that the change $a_0 \to a_0 a, a_1 \to a_1 a$ gives another homotopy between ξ and ξ'.

The simplicial homotopy itself (again as in [May92, §5]) in this case is given by maps $\alpha_0^0 \colon V_0 \to G_0$, $\alpha_i^1 \colon V_1 \to G_0 \times (G_0 \times G_1)$ for $i = 0, 1$, and $\alpha_i^2 \colon V_2 \to G_0 \times (G_0 \times G_1) \times (G_0 \times G_1 \times G_1)$, $i = 0, 1, 2$, given by

$$
\begin{aligned}
\alpha_0^0 &= y \\
\alpha_0^1 &= (d_1^* y, (x', a_0)) \\
\alpha_1^1 &= (x, (d_0^* y, a_1)) \\
\alpha_0^2 &= ((d_1 d_2)^* y, (d_2^* x', d_2^* a_1), (d_0^* x', g', {g'}^{-1} (d_2^* a_0^{-1})^{d_0^* x} g' \, d_1^* a_0)) \\
\alpha_1^2 &= (d_2^* x, ((d_0 d_2)^* y,), (d_0^* x', d_0^* a_0, d_0^* a_0^{-1} (d_2^* a_0^{-1})^{d_0^* x'} g' d_1^* a_0)) \\
\alpha_2^2 &= (d_2^* x, (d_0^* x, g), ((d_0 d_1)^* y, d_0^* a_1, d_0^* a_1^{-1} d_1^* a_1)
\end{aligned}
$$

These results are essentially the same (barring a different set of conventions) as those of [Bre90, §6.4–6.5] for the crossed module $\iota \colon G \to \mathrm{Aut}(G)$.

3.3 Bitorsor cocycles

Let $\mathscr{G}$ be a gr-stack. Let $U_\bullet$ be a hypercover, for example the Čech complex $\check{C}U$ of a generalized cover $U \to *$.

3.3.1 Definition. A *1-cocycle* with values in $\mathscr{G}$ consists of a pair (g, γ), where g is an object of $\mathscr{G}$ over U_1, and γ a morphism of $\mathscr{G}$ over U_2, satisfying the cocycle conditions

$$\gamma \colon d_1^* g \xrightarrow{\sim} d_2^* g \cdot d_0^* g \tag{3.3.2a}$$

over U_2, and the coherence condition

$$\big((d_2 d_3)^* g \cdot d_0^* \gamma\big) \circ d_2^* \gamma = a \circ \big(d_3^* \gamma \cdot (d_0 d_1)^* g\big) \circ d_1^* \gamma, \tag{3.3.2b}$$

over U_3, where a is the associator isomorphism for the group law in $\mathscr{G}$. Two cocycles (g, γ) and (g', γ') (assumed for simplicity to be defined over the same $U_\bullet$) are *equivalent* if there is a pair (h, η), where $h \in \mathrm{Ob}\,\mathscr{G}_{U_0}$ and $\eta \in \mathrm{Mor}\,\mathscr{G}_{U_1}$, such that:

$$\eta \colon g \cdot (d_0^* h) \xrightarrow{\sim} (d_1^* h) \cdot g' \tag{3.3.3a}$$

and the diagram

$$
\begin{tikzcd}
d_1^*g \cdot (d_0d_1)^*h \arrow[r, "d_1^*\eta"] \arrow[d, "\gamma"] & (d_1d_2)^*h \cdot d_1^*g' \arrow[d, "\gamma'"] \\
(d_2^*g \cdot d_0^*g) \cdot (d_0d_1)^*h \arrow[d, "a"] & (d_1d_2)^*h \cdot (d_2^*g' \cdot d_0^*g') \\
d_2^*g \cdot (d_0^*g \cdot (d_0d_1)^*h) & ((d_1d_2)^*h \cdot d_2^*g') \cdot d_0^*g' \arrow[u, "a"] \\
d_2^*g \cdot ((d_0d_2)^*h \cdot d_0^*g') \arrow[u, "d_0^*\eta"] & (d_2^*g \cdot (d_0d_2)^*h) \cdot d_0^*g' \arrow[l, "a"'] \arrow[u, "d_2^*\eta"]
\end{tikzcd}
\tag{3.3.3b}
$$

commutes.

In view of the discussion on the relationship between $\mathscr{G}$ and the crossed module reviewed in sect. 2.1, whereby the monoidal structure of $\mathscr{G}$ is described in terms of contracted products of G_1-bitorsors, a 1-cocycle such as (g, γ) in Definition 3.3.1 will be referred to, albeit imprecisely, as bitorsor cocycle.

It is easy to pass from a 1-cocycle with values in $\mathscr{G}$ to a 1-cocycle with values in $\overline{\mathrm{W}}\,\underline{G_\bullet}$. Indeed, recall from [Bre90] or from the remarks in sect. 2.1 that $\mathscr{G} \simeq \mathrm{TORS}(G_1, G_0)$, the gr-stack of G_1-torsors equipped with a chosen trivialization of their extensions to G_0. Thus $g \in \operatorname{Ob}\mathscr{G}_{U_1}$ can be thought of as such an object. In other words, we may write g as the pair $g = (E, s)$, where E is the underlying G_1-torsor and $s\colon E \to G_0$ is the equivariant morphism providing the trivialization as a G_0-torsor. So we have:

3.3.4 Lemma. *There exists a refinement $V_\bullet$ of $U_\bullet$ such that the bitorsor cocycle (g, γ) determines a 1-cocycle $V_\bullet \to \overline{\mathrm{W}}\,\underline{G_\bullet}$.*

Proof. Let $V \to U_1$ be a generalized cover such that the restriction of the underlying G_1-torsor E of $g = (E, s)$ becomes trivial. Then by [SGA72, V, Théorème 7.3.2] there exists a hypercover $V_\bullet$ and a map $V_\bullet \to U_\bullet$ which for degree $n = 1$ factorizes through the chosen cover:

$$V_1 \to V \to U_1.$$

Over V_1 we have $E|_{V_1} \simeq G_1|_{V_1}$, and s is determined by its value $s(1) \in G_0$. Thus g may simply be identified with this element of $G_0(V_1)$. In turn, the morphism γ is identified with an element of G_1 over V_2, since the underlying map of G_1-torsors is a morphism of trivial torsors. That is, the required element is simply $\gamma(1) \in G_1(V_2)$. Notice that from the identification of g with $s(1) \in G_0$ it follows that $d_2^*g \cdot d_0^*g$ is identified with the product $d_2^*s(1)d_0^*s(1)$. Since γ is a morphism of (G_1, G_0)-torsors, we must have that

$$d_1^*s(1) = d_2^*s(\gamma(1))d_0^*s(\gamma(1)) = d_2^*s(1)d_0^*s(1)\,\partial\gamma(1).$$

Furthermore, it is not difficult to realize that the coherence condition for γ on V_3 becomes

$$d_0^*\gamma(1)d_2^*\gamma(1) = d_3^*\gamma(1)^{(d_0 d_1)^* s(1)} d_1^*\gamma(1).$$

These are precisely the cocycle relations (3.2.2) (modulo exchanging $x \leftrightarrow g$ and $g \leftrightarrow \gamma$ in the notation). □

The procedure in the proof of Lemma 3.3.4 will repeatedly be used in the sequel.

3.3.5 Remark. If $U_\bullet \to \overline{\mathrm{W}}\,\underline{G}_\bullet$ is a simplicial map, where again $\underline{G}_\bullet$ is the simplicial group determined by a crossed module, a converse procedure allows one to obtain a 1-cocycle with values in the associated gr-stack $\mathscr{G}$ relative to the Čech nerve $\check{\mathrm{C}}(U_0)$, where U_0 is the degree $n = 0$ object of $U_\bullet$. The (long) proof can be extracted from [Bre90, §6.5]. No explicit use will be made of such procedure in the rest of this paper.

3.4 Torsors for gr-stacks

The definition of *torsor* under a gr-stack has been given in full generality in [Bre90, 6.1], so here we will confine ourselves to only recalling the main points. Let $\mathscr{G}$ be a gr-stack over S. In modern parlance, a $\mathscr{G}$-torsor is the categorification of the standard notion of torsor, as follows.

A right-action of $\mathscr{G}$ on a stack in groupoids $\mathscr{X}$ is given by a morphism of stacks

$$m \colon \mathscr{X} \times \mathscr{G} \longrightarrow \mathscr{X}$$

plus a natural transformation

(3.4.1)
$$\begin{tikzcd} \mathscr{X} \times \mathscr{G} \times \mathscr{G} \arrow[r, "{(m,\mathrm{id}_{\mathscr{G}})}"] \arrow[d, "{(\mathrm{id}_{\mathscr{X}},\otimes_{\mathscr{G}})}"'] & \mathscr{X} \times \mathscr{G} \arrow[d, "m"] \arrow[dl, phantom, "\Swarrow \mu"] \\ \mathscr{X} \times \mathscr{G} \arrow[r, "m"'] & \mathscr{X} \end{tikzcd}$$

which amounts, for objects x, g_0, g_1, to a functorial isomorphism

$$\mu_{x,g_0,g_1} \colon (x \cdot g_0) \cdot g_1 \xrightarrow{\sim} x \cdot (g_0 \cdot g_1),$$

where $x \cdot g$ stands for $m(x, g)$. We require that:

1. the pair (m, μ) satisfy the standard pentagon diagram;
2. the composite

$$\mathscr{X} \xrightarrow{\sim} \mathscr{X} \times \mathbf{1} \longrightarrow \mathscr{X} \times \mathscr{G} \xrightarrow{m} \mathscr{X}$$

   be isomorphic to the identity functor of $\mathrm{id}_{\mathscr{X}}$, where $\mathbf{1} \to \mathscr{G}$ sends the unique object to the identity object of $\mathscr{G}$. Moreover, this morphism must be compatible with m and μ, in the sense that the two diagrams [Bre90, (6.1.4)], resulting from combining it with (3.4.1), must be commutative.

Most importantly, we require that the morphism

$$\tilde{m} = (\mathrm{pr}_1, m)\colon \mathscr{X} \times \mathscr{G} \longrightarrow \mathscr{X} \times \mathscr{X}$$

be an equivalence. Having so far defined what ought to be called a *pseudo*-torsor, we need to complete the definition by adding the condition that there exist a (generalized) cover $U \to *$ such that the fiber category $\mathscr{X}_U$ be non-empty.

A *morphism* of $\mathscr{G}$-torsors $\mathscr{X} \to \mathscr{X}'$ consists of a stack morphism $F\colon \mathscr{X} \to \mathscr{X}'$ together with a natural transformation

$$\begin{tikzcd}
\mathscr{X} \times \mathscr{G} \arrow[r, "(F,\mathrm{Id}_{\mathscr{G}})"] \arrow[d, "m"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathscr{X} \times \mathscr{G} \arrow[d, "m'"] \\
\mathscr{X} \arrow[r, "F"'] & \mathscr{X}'
\end{tikzcd}$$

compatible with the transformations μ and μ' (That is, with the diagrams (3.4.1)).

A *2-morphism* of $\mathscr{G}$-torsors is a 2-morphism $\alpha\colon F \Rightarrow F'$ such that the diagrams

$$\begin{tikzcd}
\mathscr{X} \times \mathscr{G} \arrow[r, "(F,\mathrm{Id}_{\mathscr{G}})"] \arrow[d, "m"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathscr{X} \times \mathscr{G} \arrow[d, "m'"] \\
\mathscr{X} \arrow[r, "F"', ""{name=A}] \arrow[r, bend right, "F'"', ""{name=B}] \arrow[from=A, to=B, phantom, "\Downarrow\alpha"] & \mathscr{X}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{X} \times \mathscr{G} \arrow[r, bend left, "(F,\mathrm{Id}_{\mathscr{G}})", ""{name=C}] \arrow[r, "(F',\mathrm{Id}_{\mathscr{G}})"', ""{name=D}] \arrow[from=C, to=D, phantom, "\Downarrow\alpha"] \arrow[d, "m"'] \arrow[dr, phantom, "\swarrow\varphi'"] & \mathscr{X} \times \mathscr{G} \arrow[d, "m'"] \\
\mathscr{X} \arrow[r, "F'"'] & \mathscr{X}'
\end{tikzcd}$$

define a commutative diagram of 2-morphisms.

3.4.2 Remark. We have defined the notion of right torsors. That of left torsor is defined in the same way. It is actually the one adopted in [Bre90].

With the notions of morphism and 2-morphism outlined above, $\mathscr{G}$-torsors comprise a 2-category. In fact, all together they form a neutral 2-gerbe over S denoted $\mathrm{TORS}(\mathscr{G})$. The fiber above $U \in \mathrm{Ob}(\mathsf{S})$ is the 2-category of $\mathscr{G}|_U$-torsors (cf. [Bre92; Bre94a]).

3.5 Contracted product of gr-stacks

We will need to consider the notion of contracted product of torsors over a gr-stack in some detail. It is introduced in [Bre90, §6.7] (credited to J. Bénabou). (We use a slightly different convention for some of the diagrams.)

If $\mathscr{X}$ (resp. $\mathscr{Y}$) is a right (resp. left) $\mathscr{G}$-torsor, or more generally a stack with a $\mathscr{G}$-action, the *contracted product* $\mathscr{X} \wedge^{\mathscr{G}} \mathscr{Y}$ is defined as follows. The objects are pairs $(x, y) \in \mathrm{Ob}\,\mathscr{X} \times \mathscr{Y}$. A morphism $(x, y) \to (x', y')$ is an equivalence classes of triples (a, g, b), where $g \in \mathrm{Ob}\,\mathscr{G}$, and $a\colon x \to x' \cdot g$ and $b\colon g \cdot y \to y'$

are morphisms of $\mathscr{X}$ and $\mathscr{Y}$, respectively. Two triples (a, g, b) and (a', g', b') are equivalent if there is a morphism $\gamma\colon g \to g'$ in $\mathscr{G}$ such that the diagrams

$$\begin{tikzcd} & x' \cdot g \arrow[dd, "\mathrm{id}_{x'} \cdot \gamma"] \\ x \arrow[ur, bend left, "a"] \arrow[dr, bend right, "a'"'] & \\ & x' \cdot g' \end{tikzcd} \qquad \begin{tikzcd} g \cdot y \arrow[dd, "\gamma \cdot \mathrm{id}_y"'] \arrow[dr, bend left, "b"] & \\ & y' \\ g' \cdot y \arrow[ur, bend right, "b'"'] & \end{tikzcd}$$

commute. The composition of two morphisms $(x_1, y_1) \to (x_2, y_2)$ and $(x_2, y_2) \to (x_3, y_3)$ represented by triples (a, g, b) and (a', g', b'), respectively, is represented by the triple given by the expected compositions

$$x_1 \xrightarrow{a} x_2 \cdot g \xrightarrow{a' \cdot g'} (x_3 \cdot g') \cdot g \xrightarrow{\sim} x_3 \cdot (g' \cdot g)$$
$$(g' \cdot g) \cdot y_1 \xrightarrow{\sim} g' \cdot (g \cdot y_1) \xrightarrow{g' \cdot b} g' \cdot y_2 \xrightarrow{b'} y_3$$

and, of course, $g' \cdot g$.

It should be observed that the foregoing procedure produces a fibered category over S with group law. We denote by $\mathscr{X} \wedge^{\mathscr{G}} \mathscr{Y}$ the associated stack. One may also characterize $\mathscr{X} \wedge^{\mathscr{G}} \mathscr{Y}$ as the "2-Limit" of the diagram

$$\mathscr{X} \times \mathscr{Y} \times \mathscr{G} \rightrightarrows \mathscr{X} \times \mathscr{Y}$$

where one arrow is the projection and the other is the (right) action $(x, y, g) \to (x \cdot g, g^* \cdot y)$, where x, y, g are objects and g^* is a choice for the inverse of g.

Properties analogous to the familiar ones for ordinary torsors hold. For example, whereas in the ordinary contracted product $P \wedge^G Q$ of G-spaces one has the relation

$$(xg, y) = (x, gy),$$

namely the two pairs (xg, y) and (x, gy) represent the same point of $P \wedge^G Q$, here one has the isomorphism

$$(x \cdot g, y) \xrightarrow{\sim} (x, g \cdot y)\,,$$

represented by the triple $(\mathrm{id}_{x \cdot g}, g, \mathrm{id}_{g \cdot y})$.

3.6 Cohomology classes and classification of torsors

3.6.1 Proposition ([Bre90, Proposition 6.2])**.** *Let $G_1 \to G_0$ be a crossed module of T. The elements of the pointed set $\mathrm{H}^1(*, G_1 \to G_0)$ are in bijective correspondence with equivalence classes of right $\mathscr{G}$-torsors over S, where $\mathscr{G} = \left[G_1 \to G_0\right]^{\sim}$.*

General idea of the proof. The central argument goes through the standard computation with 1-cocycles subordinated to hypercovers $U_\bullet$. Suppose $\mathscr{X}$ is a right

$\mathscr{G}$-torsor over S, as described above. The choice of an object x of $\mathscr{X}$ over U_0 leads to establishing the existence of an object g of $\mathscr{G}$ over U_1 such that

$$d_0^* x \xrightarrow{\sim} d_1^* x \cdot g.$$

After pulling back to U_2, from the local equivalence of $\mathscr{X}$ and $\mathscr{G}$ we can conclude that there must exist a morphism (3.3.2a) over U_2, with γ satisfying (3.3.2b) over U_3. The choice of another object x' of $\mathscr{X}$, still over U_0 say, leads to another 1-cocycle (g', γ') *equivalent* to (g, γ), in the sense of Definition 3.3.1; that is, there is a pair (h, η) where h is an object of $\mathscr{G}$ over U_0 and a η morphism over U_1 satisfying equations (3.3.3).

From a 1-cocycle (g, γ) one can extract a 1-cocycle with values in the crossed module $G_1 \to G_0$ as explained at the end of sect. 3.3.

Conversely, as mentioned in Remark 3.3.5, the procedure from the proof of [Bre90, Proposition 6.2], in particular §6.5, allows us to reconstruct a bitorsor cocycle, and ultimately a $\mathscr{G}$-torsor, from a 1-cocycle with values in $G_1 \to G_0$. □

4 Pushing cohomology classes along butterflies

Changing the coefficients results in a morphism in non-abelian cohomology. From the point of view of the general definition recalled in sect. 3.1, this is done by means of a morphism of simplicial groups $\underline{H}_\bullet \to \underline{G}_\bullet$, which in our case is the one induced by a morphism of crossed modules, and ultimately by a morphism $F \colon \mathscr{H} \to \mathscr{G}$ of gr-stacks. We are also specifically interested in the case $i = 1$, and we want to provide a short account of how the morphism

$$F_* \colon \mathrm{H}^1(*, \mathscr{H}) \longrightarrow \mathrm{H}^1(*, \mathscr{G})$$

can be profitably described in terms of butterflies. This is a necessary stepping stone in the more geometric description of the first non-abelian cohomology group with values in a gr-stack to be presented further down in the paper. After some general observations, we begin with an elementary approach to the above morphism in terms of explicit 1-cocycles with values in crossed modules. We then show how the more conceptual formulation in terms of bitorsor cocycles can be reduced to these explicit calculations.

4.1 General remarks

Let $(F, \lambda) \colon \mathscr{H} \to \mathscr{G}$ be a morphism of gr-stacks over S, where we have explicitly marked the natural isomorphism λ providing the additivity:

$$\lambda_{y_1, y_2} \colon F(y_1 y_2) \xrightarrow{\sim} F(y_1) F(y_2),$$

for any two objects y_1, y_2 of $\mathscr{H}$. The following is an easy claim whose proof is left to the reader.

4.1.1 Lemma. *Let (F, λ) be as above, and let (y, h) be a 1-cocycle with values in $\mathscr{H}$ relative to a hypercover $U_\bullet \to *$ as in Definition 3.3.1. Then $(F(y), \lambda \circ F(h))$ is a 1-cocycle with values in $\mathscr{G}$ (relative to the same hypercover). If (y, h) and (y', h') are two equivalent 1-cocycles with values in $\mathscr{H}$, then so are their images $(F(y), \lambda \circ F(h))$ and $(F(y'), \lambda \circ F(h'))$.*

Our goal is to explicitly calculate $(F(y), \lambda \circ F(h))$ by means of a butterfly representing F.

4.2 Lift of a 1-cocycle along a butterfly

Since a butterfly $[H_\bullet, E, G_\bullet]$ corresponds to a morphism $F\colon \mathscr{H} \to \mathscr{G}$, it is expected that it will be possible to "lift" a 1-cocycle $\eta = (y, h)$ with values in $\overline{\mathrm{W}}\, \underline{H}_\bullet$ to one with values in $\overline{\mathrm{W}}\, \underline{G}_\bullet$. Note that, after having observed that the butterfly E or equivalently the morphism F lead to a simplicial map $\overline{\mathrm{W}}\, \underline{H}_\bullet \to \overline{\mathrm{W}}\, \underline{G}_\bullet$, the lift is only a matter of composing η with said map. We prefer to present a direct approach, which will be useful here and elsewhere in this text.

Let $V_\bullet$ be a hypercover as above, and let $\eta = (y, h)\colon V_\bullet \to \overline{\mathrm{W}}\, \underline{H}_\bullet$ be a 1-cocycle, with $y\colon V_1 \to H_0$ and $h\colon V_2 \to H_1$. Since $\pi\colon E \to H_0$ is a sheaf epimorphism, there will be a local lift of y to E, namely a (generalized) cover $p_1\colon U \to V_1$ and $e\colon U \to E$ such that

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, "p_1"'] & E \arrow[d, "\pi"] \\ V_1 \arrow[r, "y"'] & H_0 \end{tikzcd}$$

commutes. Using [SGA72, V, Théorème 7.3.2], there is a hypercover $V'_\bullet$ dominating $V_\bullet$, with a factorization $V'_1 \to U \to V_1$. All objects will be considered relative to $V'_\bullet$ by pull-back along the latter map. In particular, $\eta = (y, h)$ can now be considered as a 1-cocycle relative to $V'_\bullet$ via $V'_\bullet \to V_\bullet \to \overline{\mathrm{W}}\, \underline{H}_\bullet$.

The explicit form of the cocycle condition on (y, h), the relation $\partial_H = \pi \circ \kappa$, and the injectivity of $\imath\colon G_1 \to E$ show that there must exist $g\colon V'_2 \to G_1$ such that

$$d_1^* e = d_2^* e\, d_0^* e\, \kappa(h)\, \imath(g). \tag{4.2.1}$$

Set $x = \jmath \circ e\colon V'_1 \to G_0$. We show that the pair (x, g) determines a 1-cocycle $\xi\colon V'_\bullet \to \overline{\mathrm{W}}\, \underline{G}_\bullet$.

Applying $\jmath$ to the previous relation gives the first cocycle condition (3.2.2a). After a pull-back to V'_3, and using (4.2.1) to reduce $(d_2 d_3)^* e (d_0 d_3)^* e\, (d_0 d_1)^* e$ in both possible ways, by a routine calculation we obtain the equality

$$\kappa(d_2^* h\, d_0^* h)\, \imath(d_2^* g\, d_0^* g) = \kappa((d_3^* h)^{(d_0 d_1)^* b}\, d_1^* h)\, \imath((d_3^* g)^{(d_0 d_1)^* x}\, d_1^* g), \tag{4.2.2}$$

so that the second cocycle condition (3.2.2b) for (x, g) also holds. (This uses the fact that $\imath$ is injective and that its image commutes with that of κ.)

4.2.3 Remark. From (4.2.1) and (4.2.2), it follows that $\tilde{\eta} = (e, (h, g))$ defines a 1-cocycle with values in the crossed module $(\kappa, \imath)\colon H_1 \times G_1 \to E$.

4.2.4 Remark. The technique adopted in this section can also be used to describe the explicit lift of a 0-cocycle with values in $\overline{\mathrm{W}}\,\underline{H}_\bullet$ of the type discussed in [Part I]. It is an exercise to show that the geometric view in terms of torsors given there reduces to this one when trivializations are chosen. This view is implicit in the proof of Theorem 2.2.3 given in [Part I, Theorem 4.3.1].

4.3 Computing the map F_*

When $\mathscr{H} \xrightarrow{\sim} [H_1 \to H_0]^\sim$, $\mathscr{G} \xrightarrow{\sim} [G_1 \to G_0]^\sim$, and (F, λ) is expressed through the butterfly $[H_\bullet, E, G_\bullet]$, the image of a 1-cocycle (y, h) with values in $\mathscr{H}$ can be explicitly computed. Most of the necessary calculations follow in a straightforward way from the explicit treatment of the equivalence between the morphism F and the butterfly provided in [Part I, Theorem 4.3.1] (recalled here as Theorem 2.2.3).

Recall that we have the equivalence $\mathscr{H} \simeq \mathrm{TORS}(H_1, H_0)$, and therefore, if the object y corresponds to the (H_1, H_0)-torsor (Q, t), then $F(y)$ can be computed as

$$F(Q, t) = \underline{\mathrm{Hom}}_{H_1}(Q, E)_t\,,$$

as shown in Part I. The right-hand side is the G_1-torsor of local H_1-equivariant lifts of $t\colon Q \to H_0$ to E. In fact it is a (G_1, G_0)-torsor: the section

$$s\colon\ \underline{\mathrm{Hom}}_{H_1}(Q, E)_t \longrightarrow G_0$$

is simply the map sending a local lift e of t to $\jmath \circ e$. The morphism h is the isomorphism of torsors

$$h\colon d_1^*(Q, t) \xrightarrow{\ \sim\ } (d_2^* Q \overset{H_1}{\wedge} d_0^* Q, d_2^* t d_0^* t),$$

so that the composite $\lambda \circ F(h)$ arises, again as explained in Part I, from the isomorphism of G_1-torsors

$$\underline{\mathrm{Hom}}_{H_1}(d_2^* Q \overset{H_1}{\wedge} d_0^* Q, E)_t \xrightarrow{\ \sim\ } \underline{\mathrm{Hom}}_{H_1}(d_2^* Q, E)_t \overset{G_1}{\wedge} \underline{\mathrm{Hom}}_{H_1}(d_0^* Q, E)_t.$$

Assume the hypercover $U_\bullet$ with respect to which (y, h) is defined is such that the underlying H_1-torsor Q is trivial, and the whole cocycle can be expressed via a 1-cocycle with values in the crossed module $H_1 \to H_0$. Let us keep the notation (y, h) for the latter, so that now $y \in H_0(U_1)$ and $h \in H_1(U_2)$.

Recalling that $y \in H_0(U_1)$ corresponds to the object (H_1, y) of $\mathscr{H}(U_1)$, its image under F is given by:

$$\begin{aligned} \underline{\mathrm{Hom}}_{H_1}(H_1, E)_y &\xrightarrow{\ \sim\ } E_y \\ e &\longmapsto e(1) \end{aligned} \tag{4.3.1}$$

where the G_1-torsor on the right-hand side is the "fiber" of $E \to H_0$ above y. It follows that the resulting cocycle with values in $\mathscr{G}$ is given by the datum of E_y plus the morphism

$$\gamma\colon E_{d_1^* y} \xrightarrow{\sim} E_{d_2^* y} \overset{G_1}{\wedge} E_{d_0^* y}. \tag{4.3.2}$$

arising from the application of (F, λ) to the first relation in the 1-cocycle condition, i.e.

$$d_1^* y = d_2^* y\, d_0^* y\, \partial h,$$

which really is the morphism

$$h\colon (H_1, d_1^* y) \longrightarrow (H_1, d_2^* y\, d_0^* y).$$

So (4.3.2) is the result of the composition

$$E_{d_1^* y} \longrightarrow E_{d_2^* y\, d_0^* y} \longrightarrow E_{d_2^* y} \overset{G_1}{\wedge} E_{d_0^* y}. \tag{4.3.3}$$

A trivialization of the G_1-torsor E_y will produce a 1-cocycle with values in the crossed module $G_1 \to G_0$. More precisely, we have:

4.3.4 Proposition. *The choice of a trivialization $e \in E_y$ amounts to a lift of the 1-cocycle $\eta = (y, h)\colon U_\bullet \to \overline{\mathrm{W}}\, \underline{H}_\bullet$ along the butterfly $[H_\bullet, E, G_\bullet]$, as described in section 4.2.*

Proof. One needs to show that the choice of a trivialization $e \in E_y$ leads to formulas (4.2.1) and (4.2.2). Indeed, after pullback the choice of $e \in E_y$ yields $d_1^* e$, $d_2^* e$, and $d_0^* e$.

The first morphism of (4.3.3) sends $d_1^* e$ to $(d_1^* e)\, \kappa(h)^{-1}$. This is a consequence of the following observation: suppose we have $y = y'\, \partial h$, for $y, y' \in H_0$ and $h \in H_1$. Consider the diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_{H_1}(H_1, E)_y \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}_{H_1}(H_1, E)_{y'} \arrow[d] \\
E_y \arrow[r] & E_{y'}
\end{tikzcd}$$

where the top horizontal arrow sends a local lift e to $e \circ h^{-1}$. Then, using (4.3.1) for the vertical arrows, we can calculate the bottom horizontal arrow and find that a section e is sent by to $e\, \kappa(h)^{-1}$.

Returning to the problem at hand, since the product $d_2^* e\, d_0^* e$ provides a trivialization of $E_{d_2^* y} \wedge^{G_1} E_{d_0^* y}$, there must exist a $g \in G_1$ such that

$$(d_1^* e)\, \kappa(h)^{-1} = d_2^* e\, d_0^* e\, \imath(g),$$

which clearly is the same as (4.2.1), as wanted.

Relation (4.2.2) follows from this last one by direct calculation. Alternatively, one can show that it follows from the cocycle condition (3.3.2b) applied to the morphism (4.3.2), by pulling back to U_3 and moving from $(d_1 d_2)^* e$ to the product $(d_2 d_3)^* e\; (d_0 d_3)^* e\; (d_0 d_1)^* e$ in the two possible ways. The second approach subsumes the second. In any event, both are straightforward and left to the reader. $\square$

5 Gerbes bound by a crossed module

5.1 Recollections on gerbes

For gerbes, our main references will be [Gir71; Bre94a]. Recall that a gerbe $\mathscr{P}$ over S is by definition a stack in groupoids over S which is "locally non-empty" and "locally connected." Following [LMB00], this can be expressed as follows. Let X be a "space," i.e. a sheaf of sets, over S. A gerbe over X is a stack in groupoids $\mathscr{P}$ over S equipped with a morphism $p\colon \mathscr{P} \to X$ such that both p and the diagonal $\Delta\colon \mathscr{P} \to \mathscr{P} \times_X \mathscr{P}$ are (stack) epimorphisms. The usual definition of gerbe over S without reference to another space is recovered by setting $X = *$. Any stack $\mathscr{X}$ is equipped with a canonical morphism

$$\mathscr{X} \longrightarrow \pi_0(\mathscr{X})$$

which makes $\mathscr{X}$ into a gerbe over $\pi_0(\mathscr{X})$ ([LMB00, §3.19] and [Bre94a, §7.1]). This construction and its analog for 2-stacks were applied at different points in Part I.

If $U \to *$ is a generalized cover and G is a sheaf of groups over S/U, then $\mathscr{P}$ is a G-gerbe if there exists an object $x \in \mathrm{Ob}(\mathscr{P}_U)$ and an isomorphism

$$G \longrightarrow \underline{\mathrm{Aut}}_U(x)\,.$$

(The choice of the isomorphism is called a labeling of $\mathscr{P}$ in [Bre94a]). It is well known from loc. cit. that a G-gerbe gives rise to a non-abelian cohomology class with values in the crossed module $[\iota\colon G \to \mathrm{Aut}(G)]$. Essentially identical cohomology classes are shown in [Bre90] to arise from $\mathscr{G}$-torsors, where $\mathscr{G} = [G \to \mathrm{Aut}(G)]^\sim$ is the associated gr-stack. In fact, it is also shown in loc. cit. that there is an equivalence (of 2-gerbes) between $\mathscr{G}$-torsors and G-gerbes. This section is devoted to tie together these strands for a general crossed module $G_1 \to G_0$ of T.

5.2 Gerbes bound by a crossed module

Let $G_\bullet\colon G_1 \xrightarrow{\partial} G_0$ be a crossed module of T. The concept of gerbe bound by $G_\bullet$ is a sort of rigidification, due to Debremaeker [Deb77], of the idea of G-gerbe recalled above.

5.2.1 Definition. A gerbe $\mathscr{P}$ bound by $G_\bullet$, or equivalently, a (G_1, G_0)-gerbe, is a gerbe $\mathscr{P}$ over S equipped with the following data:

1. a functor $\mu\colon \mathscr{P} \to \mathrm{TORS}(G_0)$;
2. for each object x of $\mathscr{P}$ a functorial isomorphism $\jmath_x\colon \underline{\mathrm{Aut}}(x) \xrightarrow{\sim} \mu(x) \wedge^{G_0} G_1$ such that the diagram

(5.2.2)
$$\begin{tikzcd}
\underline{\mathrm{Aut}}(x) \arrow[r] \arrow[d, "\jmath_x"'] & \underline{\mathrm{Aut}}(\mu(x)) \arrow[d, "\simeq"] \\
\mu(x) \wedge^{G_0} G_1 \arrow[r, "\mathrm{id} \wedge \partial"] & \mu(x) \wedge^{G_0} G_0
\end{tikzcd}$$

commutes. The right vertical morphism is the standard one identifying the automorphism group of a G-torsor P with the twisted adjoint group $\operatorname{Ad} P = P \wedge^G G$.

Let us explicitly remark that the functoriality requirement made right above diagram (5.2.2) means we must have, for each morphism $f\colon x \to y$ in $\mathscr{P}$, over (say) U, a commutative diagram

(5.2.3)
$$\begin{tikzcd}
\underline{\operatorname{Aut}}(x) \arrow[r, "f_*"] \arrow[d, "\jmath_x"'] & \underline{\operatorname{Aut}}(y) \arrow[d, "\jmath_y"] \\
\mu(x) \wedge^{G_0} G_1 \arrow[r, "\mu(f)\wedge \mathrm{id}"'] & \mu(y) \wedge^{G_0} G_1
\end{tikzcd}$$

where f_* is defined, as usual, by sending a section γ of $\underline{\operatorname{Aut}}(x)$ to $f \circ \gamma \circ f^{-1}$. Furthermore, the obvious cube built from (5.2.2) and (5.2.3) should commute.

5.2.4 Example. $\mathrm{TORS}(G_1)$ is evidently a (G_1, G_0)-gerbe with $\mu = \partial_*$ and $\jmath$ given by
$$\jmath_P\colon P \overset{G_1}{\wedge} G_1 \xrightarrow{\sim} \partial_*(P) \overset{G_0}{\wedge} G_1$$
for a G_1-torsor P. $\mathrm{TORS}(G_1)$ will be called the trivial (G_1, G_0)-gerbe when equipped with the structure just described. We shall see shortly, in sect. 5.3, that all (G_1, G_0)-gerbes are locally of this type.

We will denote a gerbe bound by $G_\bullet$ synthetically as $(\mathscr{P}, \mu, \jmath)$. We have morphisms and 2-morphisms of gerbes bound by $G_\bullet$, as follows:

5.2.5 Definition. A morphism $(F, \varphi)\colon (\mathscr{P}, \mu, \jmath) \to (\mathscr{P}', \mu', \jmath')$ of gerbes bound by $G_\bullet$ is given by a morphism $F\colon \mathscr{P} \to \mathscr{P}'$ of gerbes plus a 2-morphism
$$\begin{tikzcd}
\mathscr{P} \arrow[r, "F"] \arrow[dr, bend right, "\mu"'] & \mathscr{P}' \arrow[d, "\mu'"] \\
& \mathrm{TORS}(G_0) \arrow[u, phantom, shift right=4ex, "\overset{\varphi}{\Rightarrow}"]
\end{tikzcd}$$
such that for every object $x \in \operatorname{Ob}(\mathscr{P})$ the following diagram commutes:
$$\begin{tikzcd}
\underline{\operatorname{Aut}}(x) \arrow[r, "F_*"] \arrow[d, "\jmath_x"'] & \underline{\operatorname{Aut}}(F(x)) \arrow[d, "\jmath'_{F(x)}"] \\
\mu(x) \wedge^{G_0} G_1 \arrow[r, "\varphi_x"'] & \mu'(F(x)) \wedge^{G_0} G_1
\end{tikzcd}$$
A 2-morphism $\theta\colon (E, \varepsilon) \Rightarrow (F, \varphi)$ is a 2-morphism of gerbes $\theta\colon E \Rightarrow F$ such that
$$\mu' * \theta \circ \varepsilon = \varphi.$$

In ref. [Deb77] the definition of morphism is given in greater generality than in Definition 5.2.5 above, by allowing a *strict morphism* of crossed modules. Recall that a strict morphism $f_\bullet \colon H_\bullet \to G_\bullet$ is a commutative diagram of group objects

$$\begin{tikzcd} H_1 \arrow[r, "f_1"] \arrow[d, "\partial"'] & G_1 \arrow[d, "\partial"] \\ H_0 \arrow[r, "f_0"'] & G_0 \end{tikzcd}$$

where f_1 is an f_0-equivariant map.

5.2.6 Definition. Let $(\mathscr{P}, \jmath, \mu)$ be a (G_1, G_0)-gerbe and $(\mathscr{Q}, \kappa, \nu)$ an (H_1, H_0)-gerbe. An $f_\bullet$*-morphism* $(F, \varphi) \colon \mathscr{Q} \to \mathscr{P}$ is the datum of a morphism $F \colon \mathscr{Q} \to \mathscr{P}$ of gerbes plus a 2-morphism

$$\begin{tikzcd} \mathscr{Q} \arrow[r, "F"] \arrow[d, "\nu"'] & \mathscr{P} \arrow[d, "\mu"] \\ \mathrm{TORS}(H_0) \arrow[r, "(f_0)_*"'] \arrow[ur, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathrm{TORS}(G_0) \end{tikzcd}$$

such that for each object y of $\mathscr{Q}$ there is a functorial diagram

$$\begin{tikzcd} \underline{\mathrm{Aut}}(y) \arrow[r, "\mathrm{Aut}(F)"] \arrow[d, "\kappa_y"'] & \underline{\mathrm{Aut}}(F(y)) \arrow[d, "\jmath_{F(y)}"] \\ \nu(y) \wedge^{H_0} H_1 \arrow[r, "\omega"'] & \mu(F(y)) \wedge^{G_0} G_1 \end{tikzcd}$$

where ω is the composite

$$\nu(y) \overset{H_0}{\wedge} H_1 \longrightarrow \nu(y) \overset{H_0}{\wedge} G_1 \xrightarrow{\sim} (\nu(y) \overset{H_0}{\wedge} G_0) \overset{G_0}{\wedge} G_1 \longrightarrow \mu(F(y)) \overset{G_0}{\wedge} G_1.$$

There is an obvious generalization of the notion of 2-morphism too. The reader can formulate the appropriate diagram.

5.2.7 Remark. An abelian crossed module is simply a homomorphism of abelian groups of S. Gerbes bound by crossed modules in this sense have appeared in refs. [Mil03] and [Ald08]. As it is shown in the latter, the notion encompasses several well-known examples such that of connective structure due to Brylinski and McLaughlin ([Bry93]) and hermitian structure due to one of the authors ([Ald05]).

5.3 Local description

We want to explicitly show that a (G_1, G_0)-gerbe $(\mathscr{P}, \jmath, \mu)$ is always locally equivalent to $\mathrm{TORS}(G_1)$ with the structure described in Example 5.2.4.

First, it will be useful to carry out a few local calculations to translate the global structure afforded by the (G_1, G_0)-structure on the gerbe $\mathscr{P}$ into the operations of the crossed module $\partial\colon G_1 \to G_0$. To this end, consider diagram (5.2.3), and assume two trivializations u, v of the G_0-torsors $\mu(x)$ and $\mu(y)$ are given. It follows that f determines an element $a_f \in G_0(U)$ by

$$u \longmapsto \mu(f)(u) = v\, a_f .$$

Then γ in $\underline{\mathrm{Aut}}(x)$ determines, via the trivialization u, an element $g \in G_1(U)$:

$$\jmath_x(\gamma) = u \wedge g.$$

From diagram (5.2.3) we have that the action of f_* amounts to:

$$u \wedge g \longmapsto v\, a \wedge g = v \wedge g^{a_f^{-1}} .$$

Thus, if the trivializations are fixed, the action of f_* can be identified with the automorphism of G_1 given by:

$$g \longmapsto g^{a_f^{-1}} .$$

If in particular $y = x$, so that $f \in \underline{\mathrm{Aut}}(x)$ too, then $\jmath_x(f) = u \wedge h_f$, and by (5.2.2) we must have $a_f = \partial h_f$. Since $\jmath_x(f \circ \gamma \circ f^{-1}) = u \wedge (h_f g {h_f}^{-1})$, it immediately follows that

$$g^{\partial h_f} = {h_f}^{-1}\, g\, h_f .$$

Returning to the question of the local structure of $\mathscr{P}$, let x be the choice of an object of $\mathscr{P}_U$, for a suitable $U \to *$. We can assume that there exists a trivialization s of the G_0-torsor $\mu(x)$, refining U if necessary.

5.3.1 Lemma. *The pair (x, s) determines an equivalence of (G_1, G_0)-gerbes*

$$(L_{x,s}, \lambda_{x,s})\colon \mathscr{P}|_U \xrightarrow{\ \sim\ } \mathrm{TORS}(G_1).$$

Proof. The underlying functor $L_{x,s}\colon \mathscr{P}|_U \to \mathrm{TORS}(G_1|U)$ is the standard one defined by the assignment

$$y \rightsquigarrow \underline{\mathrm{Hom}}_{\mathscr{P}}(x, y)$$

(see [Bre94a; Bre94b]). It is the choice of s that allows us to conclude that $\underline{\mathrm{Hom}}_{\mathscr{P}}(x, y)$ is a $G_1|_U$-torsor.

Let $f\colon x \to y$ be a morphism of $\mathscr{P}_U$ (over some $V \to U$) and let a be an element of G_0 over V. The claim is that the required isomorphism of $G_0|_U$-torsors

$$\lambda_{x,s}^{-1}\colon \underline{\mathrm{Hom}}_{\mathscr{P}}(x, y) \overset{G_1}{\wedge} G_0 \longrightarrow \mu(y)$$

is defined by the assignment

$$(f, a) \longmapsto \mu(f)(s)\, a.$$

Indeed, let f be replaced by $f \circ \gamma$, where γ is an automorphism of x. Then there is an element g of G_1 such that $\jmath_x(\gamma) = s \wedge g$, and by definition we have

$$\mu(\gamma)(s) = s\,\partial(g),$$

so that

$$\mu(f \circ \gamma)(s)\,a = \mu(f)(s)\,\partial(g)\,a.$$

Thus the pairs $(f \circ \gamma, a)$ and $(f, \partial(g)\,a)$ map to the same point of $\mu(y)$, hence the claim. $\square$

5.3.2 Remark. For a (G_1, G_0)-gerbe $\mathscr{P}$ choosing an object x and an appropriate trivialization of the resulting G_0-torsor $\mu(x)$ shows that $\mathscr{P}$ is in particular a G_1-gerbe.

5.4 The class of a gerbe bound by a crossed module

For gerbes bound by $G_1 \to G_0$ there is an analogous statement to Proposition 3.6.1.

5.4.1 Proposition. *The elements of the pointed set* $\mathrm{H}^1(*, G_1 \to G_0)$ *are in bijective correspondence with equivalence classes of* (G_1, G_0)*-gerbes over* S.

Proof. Let $V_\bullet \to *$ be a hypercover such that we can choose an object $x \in \operatorname{Ob} \mathscr{P}_{V_0}$ and a morphism $f\colon d_0^* x \to d_1^* x$ in $\operatorname{Mor} \mathscr{P}_{V_1}$. The choice of the pair (x, f) is a labeling of $\mathscr{P}$ relative to $V_\bullet$. Let us temporarily put $G = \underline{\mathrm{Aut}}(x)$. The computations in [Bre94b, §5.2], show that there exists an element γ of $\underline{\mathrm{Aut}}\big((d_0 d_1)^* x\big) \simeq (d_0 d_1)^* G$ over V_2, defined by the diagram

$$\begin{tikzcd}
(d_0 d_2)^* x \arrow[r, "d_2^* f"] & (d_1 d_2)^* x \\
(d_0 d_1)^* x \arrow[u, "d_0^* f"] & (d_0 d_1)^* x \arrow[l, "\gamma"'] \arrow[u, "d_1^* f"']
\end{tikzcd} \tag{5.4.2}$$

such that the non-abelian cocycle condition holds:

$$\begin{aligned}
(d_1^* f)_* &= (d_2^* f)_* \circ (d_0^* f)_* \circ (\iota_\gamma) \\
d_0^* \gamma \circ d_2^* \gamma &= \big(d_3^* \gamma\big)^{(d_0 d_1)^* f} \circ d_1^* \gamma,
\end{aligned} \tag{5.4.3}$$

where ι_γ denotes the image of $\gamma \in G$ in $\operatorname{Aut}(G)$ and γ^f is a short-hand for $(f^{-1})_*(\gamma)$. The first equation holds over V_2, whereas the second over V_3.

We can assume the $G_0|_{V_0}$-torsor $P \overset{\text{def}}{=} \mu(x)$ is trivial over some $W \to V_0$, via some choice of $s\colon W \to P$. Using [SGA72, V, Théorème 7.3.2], we can work with a new hypercover $V'_\bullet$ equipped with a map $V'_\bullet \to V_\bullet$ such that for $n = 0$ we have a factorization $V'_0 \to W \to V_0$. Let us from now on relabel $V'_\bullet$ to $V_\bullet$.

Given the foregoing assumptions, it now follows that $G = \underline{\mathrm{Aut}}(x) \simeq G_1|_{V_0}$ and f determines an element a of G_0 over V_1, whereas γ corresponds to an element g of G_1 over V_2. The local calculations of section 5.3 show that (5.4.3) becomes

$$
\begin{aligned}
(d_1^* a) &= d_2^* a \, d_0^* a \, \partial g \\
d_0^* g \, d_2^* g &= \left(d_3^* g\right)^{(d_0 d_1)^* a} d_1^* g,
\end{aligned}
\tag{5.4.4}
$$

where this time g^a denotes the action of G_0 on G_1 in the crossed module. This is a 1-cocycle in the same sense as put forward in sect. 3.2, equations (3.2.2).

The choice of a different labeling (y, f'), which for simplicity we assume to be relative to the same hypercover $V_\bullet$, will determine another pair (a', g') satisfying the same non-abelian cocycle condition (5.4.4). (To obtain it, we must assume as well that the G_0-torsor $\mu(y)$ is trivialized by an appropriate choice, possibly changing the cover again in the process.) Following [Bre94b, §5.3] we may also assume, up to further refining $V_\bullet$, that we have chosen a morphism

$$\chi \colon y \longrightarrow x$$

over V_0. Such choices determine an element η_χ of $\underline{\mathrm{Aut}}(d_0^* y)$ via

$$d_1^* \chi^{-1} \circ f \circ d_0^* \chi = f' \circ \eta_\chi .$$

Again, the calculations of section 5.3 show that the pair (χ, η_χ) determines, via the chosen trivializations, a pair (u, h), with $u \in G_0(V_0)$ and $h \in G_1(V_1)$. Combining the latter relation with the primed and unprimed versions of (5.4.3), and using (5.4.4), we arrive at the relation

$$
\begin{aligned}
a \, d_0^* u &= d_1^* u \, a' \, \partial h \\
g' \, (d_2^* h)^{d_0^* a'} d_0^* h &= d_1^* h \, g^{(d_0 d_1)^* u}.
\end{aligned}
$$

By comparison with (3.2.4), the pair (χ, η_χ) (or equivalently (u, h)) determines a homotopy between the two 1-cocycles corresponding to the two different labelings of $\mathscr{P}$.

The quickest way to reverse the process and to reconstruct a (G_1, G_0)-gerbe starting from the datum of (a, g) satisfying (5.4.4), relative to $V_\bullet$, is to follow the procedure outlined at the end of [Bre94b, §5.2]. Briefly, from a we can define a trivial (G_1, G_0)-torsor E over V_1. Now, as observed in [Bre90] and [Part I], a (G_1, G_0)-torsor is in particular a G_1-bitorsor, hence refs. [Bre94b; Bre90] may be followed to descend E (if necessary) to $V_0 \times V_0$ and then to use (5.4.4) to conclude that E defines a "bitorsor cocycle" relative to the Čech cover $\operatorname{cosk}_0 V_\bullet$, analogously to the cocycle that appeared in the proof of Proposition 3.6.1. From there, we can construct a (G_1, G_0)-gerbe by gluing local copies of $\mathrm{TORS}(G_1|_{V_0})$, considered as (G_1, G_0)-gerbes, according to Example 5.2.4. (For the gluing we must invoke the effectiveness of 2-descent data for (G_1, G_0)-gerbes.) □

5.4.5 Remark. Equations (5.4.3) and (5.4.4) in the previous proof exhibit the same triangular and tetrahedral structure as equations (3.2.2) which was made

explicit in Remark 3.2.3. After having covered the arguments in the previous proof, as well as those in the one for Proposition 3.6.1, the tetrahedral diagrams in Remark 3.2.3 should now appear natural. In particular, the labeling for the vertices reflects the choice of a trivializing object and its subsequents pullbacks along the face maps of the hypercover.

5.4.6 Remark. Embedded in the proof of the previous proposition is the fact that, given two objects $x, y \in \mathrm{Ob}\,\mathscr{P}_U$ above $U \in \mathrm{Ob}\,\mathsf{S}$, with chosen trivializations of the G_0-torsors $\mu(x)$ and $\mu(y)$, the $(\underline{\mathrm{Aut}}(x), \underline{\mathrm{Aut}}(y))$-bitorsor

$$E_{x,y} \overset{\mathrm{def}}{=} \underline{\mathrm{Hom}}_{\mathscr{P}}(y, x)$$

is in fact a (G_1, G_0)-torsor. This follows at once from the calculations of section 5.3. From this point of view an arrow $f\colon y \to x$ defined over a (generalized) cover $V \to U$ is to be considered as a local section of such torsor. In particular, the assignment defined in section 5.3 of $a_f \in G_0(V)$ to f ought to be seen as the G_1-equivariant map

$$s\colon E \longrightarrow G_0$$

which is part of the definition of (G_1, G_0)-torsor. Indeed, if f is replaced by $f \circ \gamma$, where $\gamma \in \underline{\mathrm{Aut}}(y)(V)$ and γ is then identified with an element $g \in G_1(V)$, then we have

$$a_{f\circ\gamma} = a_f\, \partial g.$$

5.5 Bitorsor cocycle associated to a labeling

According to ref. [Bre94b] and Remark 5.4.6, the proof of Proposition 5.4.1 can be reformulated in terms of the bitorsor cocycles introduced in section 3.3. Indeed, the local equivalence of (G_1, G_0)-gerbes provided by a labeling, analyzed in section 5.3, in particular in Lemma 5.3.1, determines a bitorsor cocycle as follows. Let

$$\varphi_U\colon\ \mathrm{TORS}(G_1|_U) \longrightarrow \mathscr{P}|_U$$

be such an equivalence, where $\mathscr{P}$ is a (G_1, G_0)-gerbe. Now, let U be the degree zero stage of a (generalized) cover $U_\bullet$, and consider the two possible pull-backs $d_0^*\varphi$ and $d_1^*\varphi$ to U_1. We obtain in this way a commutative diagram

$$\begin{tikzcd}
\mathrm{TORS}(G_1|_{U_1}) \arrow[rr, "\eta"] \arrow[dr, "d_1^*\varphi"'] & & \mathrm{TORS}(G_1|_{U_1}) \arrow[dl, "d_0^*\varphi"] \\
& \mathscr{P}|_{U_1} &
\end{tikzcd}$$

of (G_1, G_0)-gerbes which commutes up to natural isomorphism. By Morita theory (see [Bre94a; BM05]) η is induced by a G_1-bitorsor E. It is relatively easy to see that E is in fact an object of $\mathscr{G}_{U_1}$, that is a (G_1, G_0)-torsor over U_1. The formal argument will constitute the proof of Lemma 5.6.2 below. The pull back to U_2 determines a 2-morphism

$$\gamma\colon d_1^*\eta \Rightarrow d_2^*\eta \circ d_0^*\eta\colon\ \mathrm{TORS}(G_1|_{U_2}) \longrightarrow \mathrm{TORS}(G_1|_{U_2}),$$

which results in the morphism of bitorsors

$$\gamma\colon d_1^*E \longrightarrow d_2^*E \overset{G_1}{\wedge} d_0^*E,$$

with γ to satisfy the appropriate coherence conditions over U_3. From Lemma 3.3.4, or rather its proof, we can once again extract from (g,γ) a cocycle with values in the crossed module $G_1 \to G_0$.

5.6 Gerbes vs. torsors

Let $\mathscr{G}$ be the gr-stack $\mathrm{TORS}(G_1,G_0)$. Propositions 3.6.1 and 5.4.1 hold that $\mathscr{G}$-torsors and (G_1,G_0)-gerbes give rise to the same equivalence classes of objects, in other words they are both classified by the non-abelian cohomology set $\mathrm{H}^1(*,G_1 \to G_0)$. The following is the analog of [Bre90, Proposition 7.3] and the non-abelian counterpart of [Ald08, Theorem 5.4.4]. For the statement, recall that $\mathscr{E}q$ denotes the stack of equivalences, as defined in [Gir71, IV Proposition 5.2.5].

5.6.1 Proposition. *There is a pair of quasi-inverse Cartesian 2-functors*

$$\Phi\colon\ \mathrm{TORS}(\mathscr{G}) \longrightarrow \mathrm{GERBES}(G_1,G_0), \quad \mathscr{X} \longmapsto \mathrm{TORS}(G_1) \overset{\mathscr{G}}{\wedge} \mathscr{X}^o$$

and

$$\Psi\colon\ \mathrm{GERBES}(G_1,G_0) \longrightarrow \mathrm{TORS}(\mathscr{G}), \quad \mathscr{P} \longmapsto \mathscr{E}q(\mathrm{TORS}(G_1),\mathscr{P})$$

where for a right-$\mathscr{G}$-torsor $\mathscr{X}$ the symbol $\mathscr{X}^o$ denotes the opposite (left) torsor, which define a 2-equivalence between the 2-stacks $\mathrm{TORS}(\mathscr{G})$ *and* $\mathrm{GERBES}(G_1,G_0)$ *over* S.

In fact the pair defines a 2-equivalence between neutral 2-gerbes over S. For the proof, the following lemma, which is also of independent interest, is needed:

5.6.2 Lemma. *There is an equivalence of gr-stacks*

$$\mathscr{G} \xrightarrow{\sim} \mathscr{E}q\,(\mathrm{TORS}(G_1),\mathrm{TORS}(G_1))$$

where $\mathrm{TORS}(G_1)$ *is considered as a (G_1,G_0)-gerbe in the manner described by Example 5.2.4.*

Proof. The functor in the statement is the one sending the (G_1,G_0)-torsor (E,s) to the equivalence

$$P \longmapsto P \overset{G_1}{\wedge} E$$

where, according to [Bre90], recalled in [Part I, §3.4.8], E is a G_1-bitorsor using the left G_1-action defined as $g\cdot e = eg^{s(e)}$. The functor is clearly fully faithful.

Let $(F,\varphi)\colon\ \mathrm{TORS}(G_1) \to \mathrm{TORS}(G_1)$ be an equivalence of (G_1,G_0)-gerbes (see Definition 5.2.5). Recall that by standard arguments of Morita theory, the

underlying functor F determines and is determined, up to equivalence, by a G_1-bitorsor E so that for any right G_1-torsor P there is an isomorphism

$$F(P) \simeq P \overset{G_1}{\wedge} E.$$

E is simply the image under F of the trivial torsor G_1. By Definition 5.2.5, this must be compatible with $\partial_*\colon \mathrm{TORS}(G_1) \to \mathrm{TORS}(G_0)$, so there must exist an isomorphism

$$\varphi_P \colon P \overset{G_1}{\wedge} G_0 \overset{\sim}{\longrightarrow} \big(P \overset{G_1}{\wedge} E\big) \overset{G_1}{\wedge} G_0$$

for all torsors P. If in particular $P = G_1$, it reduces to

$$\varphi_{G_1} \colon G_0 \overset{\sim}{\longrightarrow} E \overset{G_1}{\wedge} G_0,$$

that is E must be equipped, as a right G_1-torsor, with a trivialization of its extension to a G_0-torsor. Thus E is a (G_1, G_0)-torsor, and it is relatively easy to verify that the resulting left G_1-torsor structure recalled above is the same as the original one. □

Main lines of the proof of Proposition 5.6.1. The proof closely mirrors the one in [Bre90, Proposition 7.3], except for the details pertaining to the (G_1, G_0)-gerbe structure.

By Lemma 5.6.2, $\mathscr{G}$ acts on the right on $\Psi(\mathscr{P})$. As observed in loc. cit., for any two equivalences F, F' we have $F' \simeq F \circ (F^{-1} \circ F')$, for a choice F^{-1} of the quasi-inverse to F, and $F^{-1} \circ F$ is an auto-equivalence of $\mathrm{TORS}(G_1)$. Furthermore, $\Psi(\mathscr{P})$ is locally non void, since from 5.3 the choice of an object x of $\mathscr{P}$ and of a trivialization s of $\mu(x)$ over some $U \in \mathrm{Ob}(\mathsf{S})$ determines an equivalence $\mathrm{TORS}(G_1) \overset{\sim}{\to} \mathscr{P}$ of (G_1, G_0)-gerbes over U.

As for $\Phi(\mathscr{X})$, it is a gerbe since, as already noted in loc. cit., the very fact that $\mathscr{X}$ is itself locally equivalent to $\mathscr{G}$ shows that $\Phi(\mathscr{X})$ is locally equivalent to $\mathrm{TORS}(G_1)$.

It is to be shown that $\Phi(\mathscr{X})$ actually is a (G_1, G_0)-gerbe. To this end, let $\mu\colon \Phi(\mathscr{X}) \to \mathrm{TORS}(G_0)$ be defined by

$$\mu(P, X) \overset{\text{def}}{=} \partial_*(P) = P \overset{G_1}{\wedge} G_0. \tag{5.6.3}$$

If the triple (α, g, β), where $g = (E, s)$ denotes a (G_1, G_0)-torsor, represents a morphism

$$(P_1, X_1) \longrightarrow (P_2, X_2)$$

in $\Phi(\mathscr{X})$ as described in 3.5, then $\partial_*([\alpha, g, \beta])$ is defined to be the composition

$$P_1 \overset{G_1}{\wedge} G_0 \xrightarrow{\alpha \wedge \mathrm{id}_{G_0}} (P_2 \overset{G_1}{\wedge} E) \overset{G_1}{\wedge} G_0 \overset{\sim}{\longrightarrow} P_2 \overset{G_1}{\wedge} (E \overset{G_1}{\wedge} G_0) \xrightarrow{\mathrm{id}_{P_2} \wedge s} P_2 \overset{G_1}{\wedge} G_0. \tag{5.6.4}$$

It is immediately checked that it does not depend on the specific choice of the triple representing the morphism.

For two morphisms $(P_1, X_1) \to (P_2, X_2)$ and $(P_2, X_2) \longrightarrow (P_3, X_3)$ composed as in 3.5, a diagram chase, using Mac Lane's pentagon, reveals that the composition of the corresponding images (5.6.4) equals (as expected) the image of the composition under μ.

Having defined μ, it must be proved that there is a functorial isomorphism

$$\jmath_{P,X} \colon \underline{\mathrm{Aut}}(P, X) \xrightarrow{\sim} \mu(P, X) \overset{G_0}{\wedge} G_1, \tag{5.6.5}$$

as per Definition 5.2.1. Note that from (5.6.3) it follows that:

$$\mu(P, X) \overset{G_0}{\wedge} G_1 \simeq P \overset{G_1}{\wedge} G_1 \simeq \underline{\mathrm{Aut}}(P),$$

so that (5.6.5) amounts to showing that:

$$\underline{\mathrm{Aut}}(P, X) \simeq \underline{\mathrm{Aut}}(P).$$

This actually follows from the fact that the choice of the object X of $\mathscr{X}^o$ establishes a local equivalence with $\mathscr{G}$, and hence one of $\Phi(\mathscr{X})$ with $\mathrm{TORS}(G_1)$. Explicitly, and somewhat more precisely, an automorphism of (P, X) is given by a triple (α, g, β) such that

$$\alpha \colon P \longrightarrow P \overset{G_1}{\wedge} E \qquad \beta \colon g \cdot X \longrightarrow X\,, \quad g = (E, s).$$

Since $\mathscr{X}$ is a torsor, it follows there must be an arrow

$$\gamma \colon g \longrightarrow I_{\mathscr{G}}\,,$$

in $\mathscr{G}$, that is the (G_1, G_0)-torsor (E, s) is isomorphic to the trivial (G_1, G_0)-torsor $(G_1, 1)$. It follows that the triple (α, g, β) is equivalent in the sense of 3.5 to $(\alpha', I_{\mathscr{G}}, l_X)$, where l_X is the structural functorial isomorphism

$$l_X \colon I_{\mathscr{G}} \cdot X \xrightarrow{\sim} X$$

which is part of the definition of $\mathscr{G}$-torsor. On the other hand, α' is the composition $(\mathrm{id}_P \cdot \gamma) \circ \alpha \colon P \to P \wedge^{G_1} G_1 \simeq P$, which is the sought-after element of $\underline{\mathrm{Aut}}(P)$. It is clear the requirements of Definition 5.2.1 and in 5.3 are met.

As a last point, since $\mathrm{TORS}(G_1) \wedge^{\mathscr{G}} \mathscr{X}^o$ is actually defined by a process of stackification, it should also be checked that μ as defined glues along descent data. If (P, X) is an object defined over V with a morphism

$$\varphi \colon d_0^*(P, X) \longrightarrow d_1^*(P, X)$$

over, say, $V \times_U V$ such that the cocycle condition

$$d_1^* \varphi = d_2^* \varphi \circ d_0^* \varphi$$

holds, the definition (5.6.3) should give rise to a well-defined G_0-torsor over U (via descent in $\mathrm{TORS}(G_0)$). Writing φ as being represented by a triple (α, g, β),

the descent datum above gives rise to two diagrams

$$\begin{tikzcd}
d_2^*g\cdot(d_0^*g\cdot(d_0d_1)^*X) \arrow[r] \arrow[d, "\wr"] & d_2^*g\cdot(d_0d_2)^*X \arrow[r] & (d_1d_2)^*X \\
(d_2^*g\cdot d_0^*g)\cdot(d_0d_1)^*X \arrow[rr] & & d_1^*g\cdot(d_0d_1)^*X \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
(d_0d_1)^*P \arrow[r] \arrow[d] & (d_0d_2)^*P\cdot d_0^*g \arrow[r] & ((d_1d_2)^*P\cdot d_2^*g)\cdot d_0^*g \arrow[d, "\wr"] \\
(d_1d_2)^*P & & (d_1d_2)^*P\cdot(d_2^*g\cdot d_0^*g) \arrow[ll]
\end{tikzcd}$$

Applying μ produces an object $P\wedge^{G_1}G_0$ over V and a morphism $d_0^*P\wedge^{G_1}G_0\to d_1^*P\wedge^{G_1}G_0$ of type (5.6.4) over $V\times_U V$. After having applied μ to the second diagram above, another long but totally straightforward diagram chase leads to a corresponding cocycle condition. Hence $P\wedge^{G_1}G_0$ can be descended to a G_0-torsor over U, as wanted. $\square$

Passing to classes of equivalences, we have the identifications

$$\big[\mathrm{TORS}(\mathscr{G})\big]\simeq\big[\mathrm{GERBES}(G_1,G_0)\big]\simeq \mathrm{H}^1(*,G_1\to G_0),$$

where $[\cdot]$ denotes taking classes of equivalences of objects over $*$. The first identification is of course induced by Φ (and its inverse by Ψ). It follows at once from Proposition 5.6.1 and from Propositions 3.6.1 and 5.4.1 that the above identifications constitute a commutative diagram, namely the isomorphism induced by Φ is compatible with taking cohomology classes, so that the induced map on H^1 is the identity. We record this as a lemma.

5.6.6 Lemma. *The maps induced by Φ and Ψ preserve equivalence classes.*

For future use, it is nevertheless convenient to have a computational verification.

Proof of the lemma. If $\mathscr{X}$ is a $\mathscr{G}$-torsor, then the choice of an object x in the fiber $\mathscr{X}_U$ over U establishes an equivalence

$$\mathscr{X}|_U\xrightarrow{\sim}\mathscr{G}|_U$$

which gives (see [Bre90] and the proof of Proposition 5.6.1)

$$\begin{aligned}\Phi(\mathscr{X}|_U)=\mathrm{TORS}(G_1|_U)\overset{\mathscr{G}|_U}{\wedge}\mathscr{X}^o|_U&\xrightarrow{\sim}\mathrm{TORS}(G_1|_U)\overset{\mathscr{G}|_U}{\wedge}\mathscr{G}^o|_U\\&\xrightarrow{\sim}\mathrm{TORS}(G_1|_U).\end{aligned}$$

Explicitly, an inverse equivalence is given by:

$$\begin{aligned} \varphi_U \colon \mathrm{TORS}(G_1|_U) &\xrightarrow{\sim} \mathrm{TORS}(G_1|_U) \overset{\mathscr{G}|_U}{\wedge} \mathscr{X}^o|_U \\ P &\longmapsto (P, x). \end{aligned}$$

According to section 5.5, this equivalence will determine a bitorsor cocycle for the gerbe $\Phi(\mathscr{X})$, which we want to identify with the one determined by the choice of the object x of $\mathscr{X}$. Indeed, let the latter be given by the pair (g, γ), with $g = (E, s)$ is a (G_1, G_0)-torsor over $U = U_0$, as in the proof of Proposition 3.6.1. From the morphism

$$\xi \colon d_0^* x \xrightarrow{\sim} d_1^* x \cdot g$$

in $\mathscr{X}_{U_1}$ consider the morphism (g^* is a choice of the inverse for g):

$$d_0^* x \cdot g^* \xrightarrow{\sim} \big(d_1^* x \cdot g\big) \cdot g^* \xrightarrow{\sim} d_1^* x \cdot \big(g \cdot g^*\big) \xrightarrow{\sim} d_1^* x,$$

which by definition corresponds to a morphism ξ^o in $\mathscr{X}^o$:

$$\xi^0 \colon g \cdot d_0^* x \longrightarrow d_1^* x.$$

By the definition of contracted product given in sect. 3.5, we have that the triple (id, g, ξ^o) determines a morphism

$$d_0^* \varphi\big(P \overset{G_1}{\wedge} E\big) = (P \overset{G_1}{\wedge} E, d_0^* x) \equiv (P \cdot g, d_0^* x) \xrightarrow{\sim} (P, d_1^* x) = d_1^* \varphi\big(P\big).$$

By comparison with the results of section 5.5, we see that resulting self-equivalence of $\mathrm{TORS}(G_1|_{U_1})$ is indeed given by $g = (E, s)$, as wanted.

In the opposite direction, let $\mathscr{P}$ be a (G_1, G_0)-gerbe. If x is an object of $\mathscr{P}_U$, this choice will determine as in section 5.5 a bitorsor cocycle (g, γ), relative to some cover of U, where we write again $g = (E, s)$. In view of Lemma 5.6.2, and the definition of Ψ, it is immediate that the bitorsor cocycle for the $\mathscr{G}$-torsor $\mathscr{E}q(\mathrm{TORS}(G_1), \mathscr{P})$ (relative to the trivialization induced by x) is still (g, γ). □

5.6.7 Remark. The preceding proof in fact shows that both Φ and Ψ act as identities on bitorsor cocycles, thereby implying the statement of the lemma.

6 Extension of gerbes along a butterfly

Functoriality of cohomology under a change of coefficients is one of the most important properties which are required to hold in the realm of non-abelian cohomology. In the case of groups it is well known that the map $\mathrm{H}^1(*, H) \to \mathrm{H}^1(*, G)$ induced by a homomorphism $\delta \colon H \to G$ is realized by the standard extension of torsors $\delta_* \colon \mathrm{TORS}(H) \to \mathrm{TORS}(G)$, which sends an H-torsor P to its extension $\delta_* P = P \wedge^H G$. (In fact there is a δ-morphism $P \to \delta_* P$, see [Gir71].)

In the case of a morphism $F \colon \mathscr{H} \to \mathscr{G}$ of gr-stacks, the categorification of the above extension of torsors yields the required map $\mathrm{H}^1(*, \mathscr{H}) \to \mathrm{H}^1(*, \mathscr{G})$, see ref.

[Bre90]. These matters are briefly recalled, mostly for convenience, in sect. 6.1 below. Just note that the categorification entails considering the morphism of 2-gerbes $F_*\colon \mathrm{TORS}(\mathscr{H}) \to \mathrm{TORS}(\mathscr{G})$ given by sending the $\mathscr{H}$-torsor $\mathscr{Y}$ to $F_*\mathscr{Y} = \mathscr{Y} \wedge^{\mathscr{H}} \mathscr{G}$. In view of the equivalence between torsors and gerbes stated in Proposition 5.6.1, this picture could be reinterpreted in terms of gerbes bound by crossed modules, albeit not in an immediately explicit form.

Our purpose is to remedy this by putting forward a better and more explicit picture which rests on the equivalence (cf. Theorem 2.2.3) between morphisms of gr-stacks and butterflies between crossed modules, and on the interpretation of the first non-abelian cohomology group with values in a gr-stack as equivalence classes of gerbes. The procedure to be expounded below starts with a gerbe bound by the crossed module $H_\bullet$ and uses the butterfly representing $F\colon \mathscr{H} \to \mathscr{G}$ to construct in a fairly explicit way a gerbe bound by $G_\bullet$, compatibly with the categorification above. It builds upon and improves an earlier notion of Debremaeker [Deb77].

6.1 Extension of torsors

A morphism $F\colon \mathscr{H} \to \mathscr{G}$ of gr-stacks induces a morphism

$$F_*\colon \mathrm{TORS}(\mathscr{H}) \longrightarrow \mathrm{TORS}(\mathscr{G})$$

between the corresponding 2-gerbes of torsors. The definition of F_* is the categorification of the standard "extension of the structural group" for torsors, namely if $\mathscr{Y}$ is an $\mathscr{H}$-torsor, then we define

$$F_*(\mathscr{Y}) = \mathscr{Y} \overset{\mathscr{H}}{\wedge} \mathscr{G}.$$

This was extensively used—without definition, but referring instead to [Bre90]—in Part I. Passing to cohomology, that is, to isomorphism classes of objects, it is clear that there results a corresponding maps of pointed sets:

$$\mathrm{H}^1(*, \mathscr{H}) \longrightarrow \mathrm{H}^1(*, \mathscr{G}).$$

Indeed, still according to [Bre90], this is the enabling framework to interpret the functoriality of non-abelian cohomology with values in a crossed-module. Insofar as cohomology only depends on the quasi-isomorphism class of the coefficient, and every gr-stack is equivalent to one associated to a crossed module, this covers the general case.

Let $\mathscr{H}$ and $\mathscr{G}$ be associated to crossed modules $H_1 \to H_0$ and $G_1 \to G_0$, respectively. In view of the equivalence stated in Proposition 5.6.1, there is an abstract description of F_* in terms of gerbes. Following ref. [Bre90], let us use the notation F_{**} for the morphism $\mathrm{GERBES}(H_1, H_0) \to \mathrm{GERBES}(G_1, G_0)$

resulting from F_* via the following diagram:

$$\begin{tikzcd} \mathrm{TORS}(\mathscr{H}) \arrow[r, "F_*"] \arrow[d, "\Phi"'] & \mathrm{TORS}(\mathscr{G}) \arrow[d, "\Phi"] \\ \mathrm{GERBES}(H_1, H_0) \arrow[r, "F_{**}"'] & \mathrm{GERBES}(G_1, G_0) \end{tikzcd}$$

The definition is $F_{**} = \Phi \circ F_* \circ \Psi$, and the above diagram commutes up to natural isomorphism.

It is clear that modulo the obvious isomorphism above the statement of Lemma 5.6.6, F_* and F_{**} induce the same map $\mathrm{H}^1(*, \mathscr{H}) \to \mathrm{H}^1(*, \mathscr{G})$.

Unfortunately, without additional input, F_{**} cannot be easily characterized. If $\mathscr{Y}$ is again an $\mathscr{H}$-torsor, a simple manipulation gives that the gerbe $\Phi(F_*(\mathscr{Y}))$ is equivalent to $\mathrm{TORS}(G_1) \wedge^{\mathscr{H}} \mathscr{Y}^0$, where $\mathrm{TORS}(G_1)$ carries an $\mathscr{H}$-action via

$$\mathscr{H} \xrightarrow{F} \mathscr{G} \xrightarrow{\sim} \mathscr{E}q(\mathrm{TORS}(G_1), \mathrm{TORS}(G_1)).$$

Thus, if $\mathscr{Q}$ is an (H_1, H_0)-gerbe, the previous observation suggests that its image under F_{**} is

$$F_{**}(\mathscr{Q}) = \mathrm{TORS}(G_1) \overset{\mathscr{G}}{\wedge} \Psi(\mathscr{Q})^o = \mathrm{TORS}(G_1) \overset{\mathscr{H}}{\wedge} \mathscr{E}q\,(\mathrm{TORS}(H_1), \mathscr{Q})^o.$$

To improve on this picture, we propose to provide an explicit characterization of F_{**} by employing the butterfly construction of the morphism $F\colon \mathscr{H} \to \mathscr{G}$.

6.2 Debremaeker's extension along strict morphisms

Let $f_\bullet\colon H_\bullet \to G_\bullet$ be a *strict* morphism of crossed modules, as in Definition 5.2.6. Let $(\mathscr{P}, \jmath, \mu)$ be an (H_1, H_0)-gerbe. In [Deb77], Debremaeker proved that there exists a (G_1, G_0)-gerbe $(\mathscr{P}', \jmath', \mu')$ and an $f_\bullet$-morphism $\mathscr{P} \to \mathscr{P}'$.

The gerbe $(\mathscr{P}', \jmath', \mu')$ is constructed in two steps. First, a fibered category $\mathscr{P}^*$ is defined with the same objects as $\mathscr{P}$ and morphisms given by the extension of torsors

$$\underline{\mathrm{Hom}}^*(y, x) \overset{\mathrm{def}}{=} \underline{\mathrm{Hom}}_{\mathscr{P}}(y, x) \bigwedge^{\mu(y) \wedge^{H_0} H_1} \big(\mu(y) \overset{H_0}{\wedge} G_1\big), \tag{6.2.1}$$

for any two objects x, y of $\mathscr{P}$. Note that in the above formula, to define $\mu(y) \wedge^{H_0} G_1$, G_1 is considered as an H_0 object via the homomorphism $f_0\colon H_0 \to G_0$, and that the homomorphism $\mathrm{id}_{\mu(y)} \wedge f_1\colon \mu(y) \wedge^{H_0} H_1 \to \mu(y) \wedge^{H_0} G_1$ is used for the extension. Then, the second step is to define $\mathscr{P}'$ as the stack associated to $\mathscr{P}^*$. The $f_\bullet$-morphism from $\mathscr{P} \to \mathscr{P}'$ is induced by the corresponding $\mathscr{P} \to \mathscr{P}^*$ simply given by the identity on objects and the map $f \mapsto (f, 1)$ on morphisms.

To see that $\mathscr{P}'$ is a (G_1, G_0)-gerbe, one can argue that a choice of trivializations of $\mu(y)$ and $\mu(x)$ above makes $\underline{\mathrm{Hom}}_{\mathscr{P}}(y, x)$ into an (H_1, H_0)-torsor. Consequently, $\underline{\mathrm{Hom}}^*(y, x) \simeq \underline{\mathrm{Hom}}_{\mathscr{P}}(y, x) \wedge^{H_1} G_1$ is a (G_1, G_0)-torsor. The conclusion

follows from the application of this argument to the class of $\mathscr{P}$ constructed in Proposition 5.4.1. Still according to the proposition, the modified cohomology class according to (6.2.1) is therefore the class of a (G_1, G_0)-gerbe.

To elaborate further, according to [Deb77], there is a composition

$$\underline{\mathrm{Hom}}^*(y, x) \times \underline{\mathrm{Hom}}^*(z, y) \longrightarrow \underline{\mathrm{Hom}}^*(z, x)$$

defined as follows. If γ_y is an element of $\underline{\mathrm{Aut}}(y) \simeq \mu(y) \wedge^{H_0} G_1$, and similarly for γ_z, then the composition law is defined as:

$$((f, \gamma_y), (g, \gamma_z)) \longmapsto (f \circ g, \mu(g)^{-1}(\gamma_y)\gamma_z),$$

where $\mu(g)^{-1}$ is a short-hand for the homomorphism of group objects

$$\mu(y) \overset{H_0}{\wedge} G_1 \longrightarrow \mu(z) \overset{H_0}{\wedge} G_1$$

induced by $\mu(g)^{-1} \colon \mu(y) \to \mu(z)$. Note that the functor $\mu' \colon \mathscr{P}' \to \mathrm{TORS}(G_0)$ is simply induced by the composition of μ with

$$(f_0)_* \colon \ \mathrm{TORS}(H_0) \longrightarrow \mathrm{TORS}(G_0),$$

in other words to any object x we assign $\mu(x) \wedge^{H_0} G_0$. Moreover, from (6.2.1) it immediately follows that if $y = x$ then

$$\underline{\mathrm{Aut}}^*(x) \simeq \mu(x) \overset{H_0}{\wedge} G_1 \simeq \big(\mu(x) \overset{H_0}{\wedge} G_0\big) \overset{G_0}{\wedge} G_1,$$

which gives the required isomorphism $\jmath'_x$. All the necessary requirements can be easily checked by the reader as an exercise.

It is also not hard to realize that Debremaeker's construction is actually functorial with respect to morphisms (and 2-morphisms) of (H_1, H_0)-gerbes (see [Deb77] for details). This provides us with a 2-functor

$$F^0_+ \colon \ \mathrm{GERBES}(H_1, H_0) \longrightarrow \mathrm{GERBES}(G_1, G_0) \tag{6.2.2}$$

which we seek to generalize in section 6.3, to a morphism which is not necessarily assumed to be strict.

6.2.3 Remark. The object $E_{x,y} = \underline{\mathrm{Hom}}_{\mathscr{P}}(y, x)$ is a $(\mu(x) \wedge^{H_0} H_1, \mu(y) \wedge^{H_0} H_1)$-bitorsor. It must be characterized (see again [Bre90]) by a $\mu(y) \wedge^{H_0} H_1$-equivariant morphism

$$E_{x,y} \longrightarrow \underline{\mathrm{Isom}}(\mu(x) \overset{H_0}{\wedge} H_1, \mu(y) \overset{H_0}{\wedge} H_1) \simeq \underline{\mathrm{Hom}}_{H_0}(\mu(y), \mu(x))$$

from $E_{x,y}$ considered as a right torsor. This map is simply given by

$$f \longmapsto \mu(f)^{-1} \tag{6.2.4}$$

where we use the same short-hand notation as above. Consequently, $E^*_{x,y} = \underline{\mathrm{Hom}}^*(y,x)$ given by (6.2.1) has the structure of $(\mu(x)\wedge^{H_0} G_1, \mu(y)\wedge^{H_0} G_1)$-bitorsor, since by (6.2.4) above we get an obvious map

$$\underline{\mathrm{Isom}}(\mu(x)\overset{H_0}{\wedge} H_1, \mu(y)\overset{H_0}{\wedge} H_1) \longrightarrow \underline{\mathrm{Isom}}(\mu(x)\overset{H_0}{\wedge} G_1, \mu(y)\overset{H_0}{\wedge} G_1),$$

which is equivariant with respect to

$$\mathrm{id}\wedge f_1 \colon \mu(x)\overset{H_0}{\wedge} H_1 \longrightarrow \mu(x)\overset{H_0}{\wedge} G_1.$$

According to [Bre90, Proposition 2.11], this is what is required to obtain an extension of bitorsors. Thus an alternative way to construct the gerbe $\mathscr{P}'$ is to start from the bitorsor cocycle E^* as described in [Bre94b].

6.3 Extension along a butterfly

Let now $F\colon \mathscr{H} \to \mathscr{G}$ be a general morphism of gr-stacks, and let $[H_\bullet, E, G_\bullet]$ be the corresponding butterfly (2.2.1), under the equivalence theorem 2.2.3 (we assume equivalences $\mathscr{H} \simeq [H_1 \to H_0]$ and $\mathscr{G} \simeq [G_1 \to G_0]$ have been chosen). Let also $E_\bullet\colon H_1 \times G_1 \to E$ be the intermediate crossed module, quasi-isomorphic to $H_\bullet$. Recall that there is a fraction

$$H_\bullet \xleftarrow{\sim} E_\bullet \longrightarrow G_\bullet$$

which, denoting by $\mathscr{E}$ the gr-stack associated to $E_\bullet$, factors the morphism F into

$$\mathscr{H} \longleftarrow \mathscr{E} \longrightarrow \mathscr{G},$$

where the left-pointing arrow is an equivalence. Also, let $(\mathscr{Q}, k, \nu)$ be a gerbe bound by $H_\bullet$. The following theorem generalizes the analogous statement of [Deb77, Theorem, §2, p. 66].

6.3.1 Theorem. *For a butterfly $[H_\bullet, E, G_\bullet]$ as above, and a gerbe $\mathscr{Q}$ bound by $H_\bullet$, there exists a gerbe $\mathscr{P}$ bound by $G_\bullet$. The construction of $\mathscr{P}$ is purely in terms of the butterfly $[H_\bullet, E, G_\bullet]$.*

Proof. The construction of the gerbe $\mathscr{P}$ is carried out in two steps:

- first, construct an intermediate gerbe bound by $E_\bullet$;
- second, apply the construction of sect. 6.2 to the strict morphism

(6.3.2)
$$\begin{tikzcd} H_1 \times G_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d] & G_1 \arrow[d, "\partial"] \\ E \arrow[r, "\jmath"'] & G_0 \end{tikzcd}$$

to obtain the required (G_1, G_0)-gerbe $\mathscr{P}$.

To realize the first step, let us consider the gerbe:

$$\mathscr{Q}' \overset{\text{def}}{=} \mathscr{Q} \times_{\mathrm{TORS}(H_0)} \mathrm{TORS}(E),$$

where the fiber product is of course taken in the sense of stacks: an object of $\mathscr{Q}'$ is a triple (x, f, P), where x is an object of $\mathscr{Q}$, P is an E-torsor, and f is an isomorphism

$$f \colon \nu(x) \xrightarrow{\sim} \pi_*(P) = P \overset{E}{\wedge} H_0.$$

There is an obvious morphism $\mathscr{Q}' \longrightarrow \mathscr{Q}$ given by the projection to the first factor. The proof is completed by showing that $\mathscr{Q}'$ is bound by $E_\bullet$, which we state in the following lemma. □

6.3.3 Lemma. *The gerbe $\mathscr{Q}'$ is bound by $E_\bullet \colon H_1 \times G_1 \to E$.*

Proof. Indeed, first of all there is a morphism

$$\nu' \colon \mathscr{Q}' \longrightarrow \mathrm{TORS}(E)$$

given by the projection to the second factor, and, second, there is a functorial isomorphism

$$k' \colon \underline{\mathrm{Aut}}(x, f, P) \xrightarrow{\sim} P \overset{E}{\wedge} (H_1 \times G_1) \simeq (P \overset{E}{\wedge} H_1) \times (P \overset{E}{\wedge} G_1) \tag{6.3.4}$$

satisfying the requirements in Definition 5.2.1. To see this, observe that by the very definition of stack fiber product an automorphism of (x, f, P) is given by a pair

$$\varphi \colon x \longrightarrow x \qquad \alpha \colon P \longrightarrow P$$

such that

$$\begin{tikzcd}
\nu(x) \arrow[r, "f"] \arrow[d, "\nu(\varphi)"'] & P \wedge^E H_0 \arrow[d, "\alpha \wedge \mathrm{id}"] \\
\nu(x) \arrow[r, "f"'] & P \wedge^E H_0
\end{tikzcd}$$

commutes. In other words, f determines an isomorphism

$$f_* \colon \underline{\mathrm{Aut}}(\nu(x)) \xrightarrow{\sim} \underline{\mathrm{Aut}}(P \overset{E}{\wedge} H_0)$$

so that $f_*(\nu(\varphi)) = \alpha \wedge \mathrm{id}_{H_0}$. Note that it coincides with

$$f \wedge \mathrm{id}_{H_0} \colon \nu(x) \overset{H_0}{\wedge} H_0 \longrightarrow \big(P \overset{E}{\wedge} H_0\big) \overset{H_0}{\wedge} H_0 \simeq P \overset{E}{\wedge} H_0$$

modulo the canonical isomorphism which identifies, for any G-torsor R, $\underline{\mathrm{Aut}}(R)$ with $R \wedge^G G$. Thus, the following diagram

$$\begin{tikzcd}
\underline{\mathrm{Aut}}(x) \arrow[r, "k_x"] \arrow[d, "\nu"'] & \nu(x) \wedge^{H_0} H_1 \arrow[r, "f \wedge \mathrm{id}"] \arrow[d, "\mathrm{id} \wedge \partial"'] & \big(P \wedge^E H_0\big) \wedge^{H_0} H_1 \arrow[r, "\simeq"] \arrow[d, "\mathrm{id} \wedge \partial"] & P \wedge^E H_1 \arrow[d, "\mathrm{id} \wedge \partial"] \\
\underline{\mathrm{Aut}}(\nu(x)) \arrow[r, "\simeq"] & \nu(x) \wedge^{H_0} H_0 \arrow[r, "f \wedge \mathrm{id}"] & \big(P \wedge^E H_0\big) \wedge^{H_0} H_0 \arrow[r, "\simeq"] & P \wedge^E H_0
\end{tikzcd}$$

commutes. It shows that there is an isomorphism

$$\underline{\mathrm{Aut}}(x, f, P) \xrightarrow{\sim} P \overset{E}{\wedge} H_1 \times_{(P \wedge^E H_0)} P \overset{E}{\wedge} E \simeq P \overset{E}{\wedge} \big(H_1 \times_{H_0} E\big), \tag{6.3.5}$$

and moreover, everything is clearly functorial. From the butterfly (2.2.1) it readily follows that

$$H_1 \times_{H_0} E \simeq H_1 \times G_1,$$

so that (6.3.5) is the promised isomorphism (6.3.4), and this concludes the proof of the lemma. □

6.3.6 Remark. Since the strict morphism (6.3.2) involves just the projection from $H_1 \times G_1$ to G_1, the effect of (6.2.1) is to just kill off the H_1-part of the automorphisms. More precisely, given two objects (x, f, P) and (y, g, Q) of $\mathscr{Q}'$, the torsor

$$\underline{\mathrm{Hom}}_{\mathscr{Q}'}\big((y, g, Q), (x, f, P)\big)$$

is isomorphic, via (6.3.4), to a product. In this simpler situation, the net effect of (6.2.1) is that of killing the factor relative to $P \wedge^E H_1$.

6.3.7 Remark. The construction of the gerbe $\mathscr{P}$ provided by Theorem 6.3.1 can be described by the diagram

$$\begin{tikzcd} & \mathscr{Q}' \arrow[dl] \arrow[dr] & \\ \mathscr{Q} & & \mathscr{P} \end{tikzcd}$$

Both steps in the construction of the gerbe $\mathscr{P}$ in the proof of Theorem 6.3.1 are (2-)functorial: this is clear for the first step involving the fiber product construction of the gerbe

$$\mathscr{Q}' = \mathscr{Q} \times_{\mathrm{TORS}(H_0)} \mathrm{TORS}(E)$$

bound by $E_\bullet$, and for the second step it follows from the functoriality of Debremaeker's construction itself, recalled in sect. 6.2.

Let $F \colon \mathscr{H} \to \mathscr{G}$ be the morphism of gr-stacks corresponding to the butterfly $[H_\bullet, E, G_\bullet]$. By the above, we have another 2-functor. We state it as follows:

6.3.8 Definition. Let

$$F_+ \colon \mathrm{GERBES}(H_1, H_0) \to \mathrm{GERBES}(G_1, G_0)$$

be the 2-functor given by sending the (H_1, H_0)-gerbe $\mathscr{Q}$ to its extension along the butterfly $[H_\bullet, E, G_\bullet]$.

F_+ generalizes the functor F^0_+ (see (6.2.2)), and reduces to it when F arises from a strict morphism of crossed modules. However, note that while for a strict morphism $f_\bullet \colon H_\bullet \to G_\bullet$ the resulting functor F^0_+ reviewed in section 6.2 is such that there always is an $f_\bullet$-morphism $\mathscr{Q} \to F^0_+(\mathscr{Q})$, it is not so in the current more general situation, unless one reverts to a torsor picture.

6.4 Induced map on non-abelian cohomology

We now consider the effect of F_+ on cohomology. To this end, consider the cohomology class determined by the (H_1, H_0)-gerbe $\mathscr{Q}$, and let (y, h) be a representative 1-cocycle with values in $H_\bullet$, relative to a hypercover $U_\bullet \to *$. The class of $\mathscr{P} = F_+(\mathscr{Q})$ is obtained by applying the procedure of section 4 to the class of $\mathscr{Q}$. More precisely, we have:

6.4.1 Proposition. *The lift of (y, h) along the butterfly, as described in sect. 4.2, provides a representative for the cohomology class of the (G_1, G_0)-gerbe $\mathscr{P}$ constructed in Theorem 6.3.1.*

Proof. The cocycle (y, h) is determined by the choice of an object $z \in \operatorname{Ob} \mathscr{Q}_{U_0}$, a trivialization s of the H_0-torsor $\nu(z)$, and the choice of an appropriate morphism $a \colon d_0^* z \to d_1^* z$ over U_1, see the proof of Proposition 5.4.1.

To prove the proposition, we show the lift of (y, h) along the butterfly comes from a labeling of the $(H_1 \times G_1, E)$-gerbe $\mathscr{Q}'$ provided by a pair (z', a'), where z' is an object, and $a' \colon d_0^* z' \to d_1^* z'$ a morphism, respectively mapping to z and a under the projection $\mathscr{Q}' \to \mathscr{Q}$. (The pair (z', a') determines a non-abelian 1-cocycle with values in $H_1 \times G_1 \to E$ for the gerbe $\mathscr{Q}'$.)

Only the construction of z' and a' will be carried out, leaving the details of the calculation that this indeed yields the lift of (y, h) along the butterfly to the reader. In the process, the hypercover $U_\bullet$ will need replacing with a finer one, say $U'_\bullet$, by a process we have already met several times, now, and it will be silently done without further mentioning. The need for some construction to hold "locally" will signify the need for said replacement.

The object z' can be found as follows: if $\nu \colon \mathscr{Q} \to \mathrm{TORS}(H_0)$ is the functor which is part of the (H_1, H_0)-gerbe structure of $\mathscr{Q}$, choose a (local) lift of the H_0-torsor $\nu(z)$ to an E-torsor P, so that there is a π-morphism of torsors

$$\sigma \colon P \longrightarrow \nu(z), \tag{6.4.2}$$

where $\pi \colon E \to H_0$. Then set $z' = (z, f, P)$, where f is the inverse of the morphism induced by σ:

$$\begin{aligned} \bar{\sigma} \colon P \overset{E}{\wedge} H_0 &\longrightarrow \nu(z) \\ (p, y) &\longmapsto \sigma(p)\, y. \end{aligned}$$

A morphism $a' \colon d_0^* z' \to d_1^* z'$ mapping to $a \colon d_0^* z \to d_1^* z$ under the projection $\mathscr{Q}' \to \mathscr{Q}$ is of the form $a' = (a, \alpha)$, where $\alpha \colon d_0^* P \longrightarrow d_1^* P$. In fact α can be constructed as a (local) lift of $\nu(a)$ with respect to the π-morphism (6.4.2), so that we have a commutative diagram

$$\begin{tikzcd} d_0^* P \arrow[r, "\alpha"] \arrow[d, "d_0^* \sigma"'] & d_1^* P \arrow[d, "d_1^* \sigma"] \\ d_0^* \nu(z) \arrow[r, "\nu(a)"'] & d_1^* \nu(z) \end{tikzcd} \tag{6.4.3}$$

as follows. Choose $\tilde{s}$ of P such that $\sigma(\tilde{s}) = s$, again changing $U_\bullet$ if necessary. Indeed, note that the "fiber" $P_s = \sigma^{-1}(s)$ is a G_1-torsor, so finding $\tilde{s}$ amounts to a trivialization of P_s. Let $e \in E(U_1)$ be a local lift of $y \in H_0(U_1)$ and define α as:

$$\alpha(d_0^* \tilde{s}) = (d_1^* \tilde{s})\, e.$$

Since y is determined by the relation $\nu(a)(d_0^* s) = (d_1^*)y$, it is clear that α so defined satisfies (6.4.3).

Now, a further pull-back to U_2 determines an automorphism η' of $(d_0 d_1)^* z'$ such that

$$d_1^* a' = d_2^* a' \circ d_0^* a' \circ \eta' \tag{6.4.4}$$

via the analog of diagram (5.4.2) in the proof of Proposition 5.4.1. By construction, the projection $\mathscr{Q}' \to \mathscr{Q}$ maps η' to the automorphism η of $(d_0 d_1)^* z$ obtained in the same way from $a\colon d_0^* z \to d_1^* z$. It follows that $\eta' = (\eta, \varepsilon)$, where ε is an automorphism of $(d_0 d_1)^* P$ covering $\nu(\eta)$. By using (6.3.5) we have that

$$\underline{\mathrm{Aut}}((d_0 d_1)^* z' \xrightarrow{\sim} (d_0 d_1)^* P \overset{E}{\wedge} (H_1 \times_{H_0} E),$$

so that, relative to the chosen a trivialization $\tilde{s}$ of P (suitably pulled back to U_2), η' is identified with an element of $H_1 \times_{H_0} E$. In particular, ε is identified with the E-factor, call this particular element $e' \in E(U_2)$, whereas the H_1 factor is $h \in H_1(U_2)$, which corresponds to η via the chosen trivialization s of $\nu(z)$. So, explicitly, the pair (h, e') satisfies $\partial(h) = \pi(e')$. Finally, the isomorphism $H_1 \times_{H_0} E \simeq H_1 \times G_1$, identifies (h, e') with (h, g), for a suitable $g \in G_1(U_2)$, or put it differently, $e = \kappa(h)\, \imath(g)$.

Calculating the relation (6.4.4) with respect to the chosen trivializations s and $\tilde{s}$, we find that e, h, and g satisfy

$$d_1^* e = d_2^* e\, d_0^* e\, \kappa(h)\, \imath(g),$$

which is the same as (4.2.1). Moreover, from the second relation of (5.4.3) applied to the pair (a', η'), or alternatively performing the calculation suggested at the end of 4.2, it follows that e, h, and g also satisfy (4.2.2), and so the 1-cocycle (x, g), where $x = \jmath(e)$, is the lift of (y, h) along the butterfly, as wanted.

To complete the proof, we must make sure (x, g) indeed is the 1-cocycle arising from a labeling of the gerbe $\mathscr{P}$, obtained from $\mathscr{Q}'$ via the strict morphism $E_\bullet \to G_\bullet$. This is clear, since from section 6.2 we have that $\mathscr{P}$ has locally the same objects as $\mathscr{Q}'$, the functor $\mu\colon \mathscr{P} \to \mathrm{TORS}(G_0)$ is locally the composition of ν' with $\jmath_*\colon \mathrm{TORS}(E) \to \mathrm{TORS}(G_0)$, and the automorphism group of an object is locally isomorphic to G_1 via

$$H_1 \times_{H_0} E \simeq H_1 \times G_1 \longrightarrow G_1.$$

□

It follows from the previous proposition and from the arguments in section 4 that the class gerbe $\mathscr{P}$ is therefore the image of that of $\mathscr{Q}$ under F. The following is an immediate consequence of the previous results.

6.4.5 Theorem. *The gerbe $\mathscr{P}$ constructed in Theorem 6.3.1 is equivalent to $F_{**}(\mathscr{Q})$. The two 2-functors F_{**} and F_{+} are equivalent.*

7 Commutativity conditions

The group law of a gr-stack may be equipped with commutativity constraints. Cohomology with values in such a gr-stack will inherit corresponding structures, actually in a more rigid form due to the process of modding out by the relation generated by (functorial) equivalence. Butterflies help one to obtain explicit forms for these structures. (Commutativity conditions for gr-stacks are thoroughly discussed [Bre94a; Bre99], see also the discussion in [Part I, §7].)

7.1 Commutativity conditions and butterflies

The very first commutativity condition one may impose on a gr-stack is that the group law[1]

$$m\colon \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G} \tag{7.1.1}$$

be *braided,* that is that there be a functorial isomorphism

$$s_{x,y}\colon x\, y \longrightarrow y\, x$$

for each pair of objects x, y of $\mathscr{G}$. Following the convention adopted in [Part I] (which is not the same as refs. [Bre94a; Bre99]) we say that the braiding is *symmetric* if for all pairs of objects x, y of $\mathscr{G}$ the additional condition

$$s_{y,x} \circ s_{x,y} = \mathrm{id}_{x\, y}$$

holds. In addition the symmetric braiding is *Picard* if it satisfies

$$s_{x,x} = \mathrm{id}_{x\, x}$$

for each object x. A braiding is equivalent to the group law being a *morphism of gr-stacks,* rather than just a morphism of the underlying stacks, which is the categorical analogue of the very well-known fact that a group is abelian if and only if its multiplication map is a group homomorphism. Therefore there is a butterfly

$$\begin{tikzcd}
G_1 \times G_1 \arrow[dd, "\partial\times\partial"'] \arrow[dr, "\alpha"] & & G_1 \arrow[dl, "\beta"'] \arrow[dd, "\partial"] \\
& P \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\
G_0 \times G_0 & & G_0
\end{tikzcd} \tag{7.1.2}$$

[1] We are going to use a plain symbol m to denote the monoidal structure of $\mathscr{G}$, in place of the forbidding $\otimes_{\mathscr{G}}$ used in [Part I].

representing the morphism (7.1.1), see [Part I, 7.1.3], once an equivalence $\mathscr{G} \simeq [G_1 \to G_0]^\sim$ has been chosen. This particular butterfly has certain additional properties, in particular it is always *strong,* namely it always possesses a global set-theoretic section τ of the epimorphism $\rho\colon P \to G_0 \times G_0$, so that a classical braiding map ([JS93])

$$c\colon G_0 \times G_0 \longrightarrow G_1$$

can be obtained, see [Part I, §7.1]. The group law of P can then be described explicitly in terms of the set-theoretic isomorphism $P \xrightarrow{\sim} G_0 \times G_0 \times G_1$ determined by τ and the braiding.

Depending on whether the braiding is symmetric or Picard, the butterfly (7.1.2) satisfies extra symmetry conditions, described in detail in [Part I, §7]. Briefly, if $\mathscr{G}$ is braided symmetric the corresponding butterfly (7.1.2) has the property that its pull-back under the map that swaps the two factors in $G_\bullet \times G_\bullet$ is isomorphic to P. If in addition $\mathscr{G}$ is Picard, then the pull-back of this isomorphism to the diagonal is the identity.

7.2 The monoidal 2-stack of $\mathscr{G}$-torsors

Let $\mathscr{G}$ be at least braided. Since the monoidal structure of $\mathscr{G}$ is a morphism of gr-stacks, we obtain a 2-functor:

$$m_*\colon \mathrm{TORS}(\mathscr{G}) \times \mathrm{TORS}(\mathscr{G}) \longrightarrow \mathrm{TORS}(\mathscr{G}) \tag{7.2.1}$$

where we have used the identification $\mathrm{TORS}(\mathscr{G} \times \mathscr{G}) \simeq \mathrm{TORS}(\mathscr{G}) \times \mathrm{TORS}(\mathscr{G})$. Thus, m_* assigns to the $\mathscr{G} \times \mathscr{G}$-torsor $(\mathscr{X}, \mathscr{X}')$ the $\mathscr{G}$-torsor $(\mathscr{X}, \mathscr{X}') \wedge^{\mathscr{G} \times \mathscr{G}} \mathscr{G}$.

By the theory of section 6.3 the gerbe counterpart of (7.2.1) is the 2-functor

$$m_+\colon \mathrm{GERBES}(G_1, G_0) \times \mathrm{GERBES}(G_1, G_0) \longrightarrow \mathrm{GERBES}(G_1, G_0) \tag{7.2.2}$$

given by the lift of the gerbe $(\mathscr{P}, \mathscr{P}')$ along the butterfly (7.1.2).

A full investigation of the monoidal structure (7.2.1) or (7.2.2) is beyond the scope of the present work, but it is necessary to at least point out that it is the entire collection (in this case: 2-gerbe) of geometric objects itself that acquires a (weak) group structure. The one on cohomology is then obtained by considering equivalence classes, and it is examined in the next section.

7.3 Group structures on cohomology and butterflies

If $\mathscr{G}$ is at least braided, its monoidal structure (7.1.1) induces morphisms

$$m_*\colon \mathrm{H}^i(*, \mathscr{G}) \times \mathrm{H}^i(*, \mathscr{G}) \longrightarrow \mathrm{H}^i(*, \mathscr{G}), \tag{7.3.1}$$

by the mechanisms expounded both in [Part I] (for degree $i \leq 0$) and in the present work (for degree $i = 0, 1$). The morphism (7.3.1) is obtained starting from either (7.2.1) or (7.2.2) and using functoriality.

At the level of representing cocycles, the group laws (7.3.1) can be computed by applying the lifting along the butterfly (7.1.2) described in section 4.2 (By

the observation in remark 4.2.4, it applies equally well to 0-cocycles, i.e. descent data for objects of gr-stacks). The weak form of the group law for $\mathscr{G}$ translates into a standard rigid one for the m_*, including the case $i = 1$.

We collect the main facts in the following

7.3.2 Proposition. *Let $\mathscr{G}$ be a braided gr-stack.*

1. *$\mathrm{H}^0(*, \mathscr{G})$ is an abelian group;*
2. *$\mathrm{H}^1(*, \mathscr{G})$ is a group;*
3. *If in addition $\mathscr{G}$ is symmetric, $\mathrm{H}^1(*, \mathscr{G})$ is an abelian group.*

Sketch of the proof. The result is quite well-known, so we only sketch the main ideas.

For 1, given that $\mathrm{H}^0(*, \mathscr{G}) \simeq \pi_0(\mathscr{G}(*))$, the result is obvious (it follows immediately from the weak group law of $\mathscr{G}$). As noted, for case 2, that is $\mathrm{H}^1(*, \mathscr{G})$, it follows from either morphism in section 7.2 and functoriality.

More interesting is the case of a symmetric gr-stack. It was proved in [Part I, Propositions 7.2.2 and 7.2.3] that the symmetry condition is equivalent to the braiding being a *2-morphism*

$$s\colon m \Longrightarrow m \circ T\colon \mathscr{G} \times \mathscr{G} \longrightarrow \mathscr{G}$$

of gr-stacks, where T is the swap functor. Passing to cohomology and using (7.3.1) yields the commutative structure

$$\begin{tikzcd}
\mathrm{H}^1(*, \mathscr{G}) \times \mathrm{H}^1(*, \mathscr{G}) \arrow[r, "\sim"] \arrow[d, "T_*"'] & \mathrm{H}^1(*, \mathscr{G} \times \mathscr{G}) \arrow[r, "m"] \arrow[d, "T_*"'] & \mathrm{H}^1(*, \mathscr{G}) \arrow[d, equal] \\
\mathrm{H}^1(*, \mathscr{G}) \times \mathrm{H}^1(*, \mathscr{G}) \arrow[r, "\sim"] & \mathrm{H}^1(*, \mathscr{G} \times \mathscr{G}) \arrow[r, "m"] & \mathrm{H}^1(*, \mathscr{G})
\end{tikzcd}$$

□

7.4 Explicit cocycles

Besides "explaining" how the first non-abelian cohomology group with values in a crossed module acquires a group structure, with the butterfly we can calculate explicit formulas for the product. The computations involved are tedious and straightforward overall, so we will not dwell on the details and only report the main formulas.

As already observed the butterfly (7.1.2) is strong, so the group law of P can be explicitly described in terms of the set-theoretic isomorphism $P \simeq G_0 \times G_0 \times G_1$ and the braiding c as

$$(x_0, y_0, g_0)\,(x_1, y_1, g_1) = (x_0x_1, y_0y_1, c(x_1, y_0)^{y_1} g_0^{y_0y_1} g_1),$$

with $x_0, x_1, y_0, y_1 \in G_0$, and $g_0, g_1 \in G_1$. In the foregoing the strong set-theoretic section $\tau\colon G_0 \times G_0 \to P$ is obviously of the form

$$\tau(x, y) = (x, y, 1),$$

with $x, y \in G_0$. In fact, all the maps in (7.1.2) have explicit descriptions in these coordinates, and their form will be left as an exercise to the interested reader; here we only mention that $\sigma\colon P \to G_0$ has the form

$$\sigma(x, y, g) = x\, y\, \partial g.$$

Note that the composition with τ gives the multiplication map of G_0, which is of course not a homomorphism.[2] The two main computations are as follows.

Degree zero

Assume two global objects $X, X' \in \operatorname{Ob} \mathscr{G}(*)$ are represented by zero-cocycles (descent data) (x, g) and (x', g') relative to some common (hyper)cover $U_\bullet \to *$. Here $x, x' \in G_0(U_0)$ and $g, g' \in G_1(U_1)$. The object (X, X') of $\mathscr{G} \times \mathscr{G}$ is represented by the direct product of the corresponding cocycles. Applying the procedure of section 4.2 (adapted to 0-cocycles, as per Remark 4.2.4) one finds that the image of (X, X') under the multiplication map (7.3.1) is represented by the cocycle

$$(xx', g^{d_1^* x} g').$$

This formula coincides with the one for the group law of the gr-stack $\mathscr{G}$ expressed in terms of descent data found in [Part I, 3.4.3]. So the lift along the butterfly computes exactly the same (abelian) group law as induced by the braided structure on $\mathscr{G}$.

7.4.1 Remark. A priori there appear to be *two* group laws on $\mathrm{H}^0(*, \mathscr{G})$. One inherited from the monoidal structure of $\mathscr{G}$, while the second is m_* in (7.3.1). One is a homomorphism of the other, so by the classical argument they coincide, and the resulting structure is abelian.

Degree one

Assume now $\mathscr{P}, \mathscr{P}'$ are two gerbes bound by the crossed module $G_\bullet$. Recycling symbols, assume they are represented by 1-cocycles (x, g) and (x', g') relative to some common (hyper)cover $U_\bullet \to *$. This time $x, x' \in G_0(U_1)$ and $g, g' \in G_1(U_2)$. The product gerbe $\mathscr{P} \times \mathscr{P}'$ is represented by the direct product of the corresponding cocycles. Applying again the procedure of section 4.2 the gerbe $m_+(\mathscr{P} \times \mathscr{P}')$ of section 6.3 (see in particular Definition 6.3.8) is represented by a 1-cocycle relative to $U_\bullet$ given by the expression:

$$\big(x\, x', c(d_0^* x, d_2^* x')^{-d_0^* x'} g^{d_2^* x'\, d_0^* x'} g'\big). \tag{7.4.2}$$

[2] In this way one arrives at the standard interpretation of the braiding map as the isomorphism relating the multiplication map and its swapped version.

We could have used $\mathscr{G}$-torsors $\mathscr{X}$ and $\mathscr{X}'$ to arrive at the same conclusion. In particular, if (x,g) and (x',g') are assumed to be 1-cocycles corresponding to $\mathscr{X}$ and $\mathscr{X}'$, then the 1-cocycle of expression (7.4.2) represents the $\mathscr{G}$-torsor $(\mathscr{X}\times\mathscr{X}')\wedge^{\mathscr{G}\times\mathscr{G}}\mathscr{G}$.

In summary, modulo the appropriate notion of equivalence, expression (7.4.2) gives an explicit form to the group law (7.3.1) when $i=1$.

If $\mathscr{G}$ is braided symmetric, the geometric condition on the butterfly (7.1.2) translates into the standard notion that the braiding map satisfies the symmetry condition $c(x,y)=c(y,x)^{-1}$. In this instance it is possible to explicitly verify that $\mathrm{H}^1(*,\mathscr{G})$ becomes an abelian group; exchanging the role of (x,g) and (x',g') in expression (7.4.2) leads to a 1-cocycle which can be seen to be equivalent to the original one. We omit the details.

8 Butterflies and extensions

Group extensions and non-abelian cohomology in degree one have a close relationship, which one can trace from Dedeker's classical approach based on cocycle calculations, to Grothendieck's and Breen's more geometric one, where the category of extensions

$$1\longrightarrow G\longrightarrow E\longrightarrow\Gamma\longrightarrow 1$$

of the topos T is given geometric meaning by showing its equivalence to that a morphism of gr-stacks

$$\Gamma\longrightarrow \mathrm{BITORS}(G).$$

$\mathrm{BITORS}(G)$ is the gr-stack associated to the crossed module $G\to\mathrm{Aut}(G)$, and Γ is considered as a gr-stack in the obvious way. These ideas fit very well within the butterfly framework.

8.1 The Schreier-Grothendieck-Breen theory of extensions

Following ref. [Bre90, §8.11], consider an *extension of* Γ *by the crossed module* $G_1\to G_0$, a notion due to Dedecker and defined by the following commutative diagram:

(8.1.1)
$$\begin{tikzcd} 1 \arrow[r] & G_1 \arrow[r, "\imath"] \arrow[d, "\partial"'] & E \arrow[r, "\pi"] \arrow[dl, bend left, "\jmath"] & \Gamma \arrow[r] & 1 \\ & G_0 & & & \end{tikzcd}$$

where the map $\jmath\colon E\to G_0$ is subject to the additional condition

(8.1.2)
$$e^{-1}\imath(g)e=\imath(g^{\jmath(e)}).$$

We recognize (8.1.2) as the first relation in (2.2.2), as well as [Bre90, equation (8.11.2)], after the obvious changes due to the different conventions adopted in this paper.

The trivial extension corresponds to $E = \Gamma \ltimes G_1$, where Γ acts on G_1 via a homomorphism $\xi\colon \Gamma \to G_0$ and the action of G_0 on G_1, whereas $\jmath$ is given set-theoretically as

$$\jmath(x, g) = \xi(x)\, \partial g,$$

for $x \in \Gamma$ and $g \in G_1$.

A comparison with diagram (2.2.1) suggests diagram (8.1.1) ought to be considered as a "one-winged butterfly," namely a butterfly diagram from the crossed module $[1 \to \Gamma]$ to $[G_1 \to G_0]$. Therefore, by the results in [Part I, §4 and §5], recalled in section 2.2, the extension (8.1.1) corresponds to a morphism of gr-stacks

$$F_E\colon \Gamma \longrightarrow \mathscr{G}$$

where $\mathscr{G} \simeq [G_1 \to G_0]^{\sim}$. The form of this morphism is as follows. If $x\colon U \to \Gamma$ is a point, it follows from [Part I, §4.3] (see also section 4.3 for a quick review), that it maps to the $(G_1|_U, G_0|_U)$-torsor

$$\underline{\mathrm{Hom}}_1(1, E)_x \simeq x^* E \equiv E_x.$$

This retrieves the expression [Bre90, 8.2.2]. Observe also that (8.1.2) is none other than the expression of the left G_1-action on x^*E in terms of the right one (cf. section 2.1). In this language a trivial extension corresponds to a split butterfly. Note also that for a split extension the $(G_1|_U, G_0|_U)$-torsor x^*E is isomorphic to $(G_1|_U, x)$.

The obvious notion of morphism of extensions of the form (8.1.1) is clearly the same as that of morphism of one-winged butterflies, in other words an isomorphism $\varphi\colon E \to E'$ of group objects compatible with (8.1.1). With reference to the notation used elsewhere in this series (see, e.g. section 2.2) we have

$$\mathsf{Ext}(\Gamma, G_1 \to G_0) \equiv \mathsf{B}(\Gamma, G_\bullet),$$

where the left-hand side denotes the category (in fact, the groupoid) of extensions of the form (8.1.1), and the right-hand side the one of butterflies. It immediately follows from Theorem 2.2.3 that there is an equivalence of categories

$$\mathsf{Ext}(\Gamma, G_1 \to G_0) \xrightarrow{\ \sim\ } \mathsf{Hom}(\Gamma, \mathscr{G}). \tag{8.1.3}$$

There is also the fibered analog of the preceding construction. Again from [Part I, §4 and §5] (see also the summary in section 2.3), and using the same notation, we obtain the following analog of [Bre90, Lemme 8.3]:

8.1.4 Lemma. *There is an equivalence*

$$\mathscr{E}xt(\Gamma, G_1 \to G_0) \xrightarrow{\ \sim\ } \mathscr{H}om(\Gamma, \mathscr{G}),$$

where the left-hand side is the stack whose fiber over U is $\mathsf{Ext}(\Gamma|_U, G_\bullet|_U)$.

The cohomological classification of the extensions is obtained by applying π_0 to (8.1.3),

$$\mathrm{Ext}(\Gamma, G_1 \to G_0) \xrightarrow{\sim} \mathrm{Hom}(\Gamma, \mathscr{G}),$$

and rephrasing the right-hand side in terms of the non-abelian cohomology of the classifying object $\mathrm{B}\Gamma$. Briefly, the group structure of Γ is encoded by diagram 8.1.2 of [Bre90], which we write in the form

$$\gamma\colon d_1^* E \xrightarrow{\sim} d_2^* E \overset{G_1}{\wedge} d_0^* E, \tag{8.1.5}$$

subject to the coherence condition for γ expressing the associativity of the group law. Pulling back by $x\colon U \to \Gamma$, and then $d_0^* x$, $d_1^* x$, $d_1^* x$, we can see (8.1.5) plus the coherence condition for γ define a 1-cocycle on $\mathrm{B}\Gamma$ with values in $\mathscr{G}$. By a reasoning entirely analogous to the one of section 4.3, we can compute the class with values in the crossed module $G_\bullet$, thereby obtaining the sought-after element in $\mathrm{H}^1(\mathrm{B}\Gamma, \mathscr{G})$. Thus we have:

8.1.6 Proposition (Bre90, Proposition 8.2)**.** *There is a functorial isomorphism of sets*

$$\mathrm{Ext}(\Gamma, G_1 \to G_0) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{B}\Gamma, \mathscr{G}).$$

Functoriality is built into the butterfly representation of morphisms of gr-stacks.

8.2 Remarks on extensions by commutative crossed modules

We can combine the idea of extension by a crossed module (8.1.1) with the conditions studied in section 7. In this situation the first non-abelian cohomology set $\mathrm{H}^1(\mathrm{B}\Gamma, \mathscr{G})$ acquires a group structure, possibly abelian if $G_\bullet$ is symmetric or Picard.

Baer sums

The explicit cocycle multiplication formula (7.4.2) could easily be translated in terms of group cohomology. This is easier in the case of a strong butterfly, that is for an extension (8.1.1) possessing a global set-theoretic section $s\colon \Gamma \to E$, and it is left as an exercise to the reader.

There is a more interesting "butterfly explanation" of the existence of the product; while the basic mechanism is the one already explained in section 7, the translation in terms of group cohomology gives it a slightly different flavor that further underscores the role of butterfly diagrams. The procedure outlined below is the analog in the context of non-abelian cohomology of the standard Baer sum of extensions in ordinary homological algebra (see [ML95]).

From two extensions of type (8.1.1), we can form the direct product (drawn with a different orientation) one-winged butterfly:

(8.2.1)
$$
\begin{tikzcd}
 & & G_1 \times G_1 \arrow[dl, "(\imath,\imath')"'] \arrow[dd, "(\partial,\partial)"] \\
 & E \times E' \arrow[dl, "(\pi,\pi')"'] \arrow[dr, "(\jmath,\jmath')"] & \\
\Gamma \times \Gamma & & G_0 \times G_0
\end{tikzcd}
$$

which then can be composed with (7.1.2), which encodes the monoidal structure, to yield

$$
\begin{tikzcd}
 & & G_1 \times G_1 \arrow[dl, "(\imath,\imath)"'] \arrow[dd, "(\partial,\partial)"] \arrow[dr, "\alpha"] & & G_1 \arrow[dl, "\beta"'] \arrow[dd, "\partial"] \\
 & E \times E' \arrow[dl, "(\pi,\pi)"'] \arrow[dr, "(\jmath,\jmath)"] & & P \arrow[dl, "\rho"'] \arrow[dr, "\sigma"] & \\
\Gamma \times \Gamma & & G_0 \times G_0 & & G_0
\end{tikzcd}
$$

that is, according to [Part I, §5.1],

$$
\begin{tikzcd}
 & & G_1 \arrow[dl] \arrow[dd, "\partial"] \\
 & \left(E \times E'\right) \times^{G_1 \times G_1}_{G_0 \times G_0} P \arrow[dl] \arrow[dr] & \\
\Gamma \times \Gamma & & G_0
\end{tikzcd}
$$

which is then pulled back to Γ via the diagonal homomorphism $\Delta\colon \Gamma \to \Gamma \times \Gamma$. The overall picture for the product is as follows:

$$
\begin{tikzcd}
1 \arrow[r] \arrow[dd] & 1 \arrow[dd] \arrow[dr] & & G_1 \arrow[dl] \arrow[dd, "\partial"] \\
 & & \left(E \times E'\right) \times^{G_1 \times G_1}_{G_0 \times G_0} P \arrow[dl] \arrow[dr] & \\
\Gamma \arrow[r, "\Delta"] & \Gamma \times \Gamma & & G_0
\end{tikzcd}
$$

The composition expressed by the above diagram is the full butterfly diagram expressing the product structure on the first cohomology with coefficients in $\mathscr{G}$. Thus we obtain a monoidal structure on the category $\mathsf{Ext}(\Gamma, G_\bullet)$.

Abelian structure on H^1

If $\mathscr{G}$ (or equivalently $G_\bullet$) is symmetric, the butterfly (7.1.2) is isomorphic to itself under pull-back by the morphism T that switches the factors. By [Part I, §7.2.4] this means there exists $\psi\colon P \xrightarrow{\sim} P$ such that:

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d] & P \arrow[d] \\ G_0 \times G_0 \arrow[r, "T"] & G_0 \times G_0 \end{tikzcd}$$

compatible with all the morphisms in (7.1.2). The same kind of swap of course exchanges the factors in the butterfly (8.2.1). Therefore there is a diagram of juxtaposed butterflies

$$\begin{tikzcd} \Gamma \arrow[r, "\Delta"] \arrow[d, equal] & \Gamma \times \Gamma \arrow[d, "T"] & E \times E' \arrow[l] \arrow[r] \arrow[d, "T"] & G_\bullet \times G_\bullet \arrow[d, "T"] & P \arrow[l] \arrow[r] \arrow[d, "\psi"] & G_\bullet \arrow[d, equal] \\ \Gamma \arrow[r, "\Delta"] & \Gamma \times \Gamma & E' \times E \arrow[l] \arrow[r] & G_\bullet \times G_\bullet & P \arrow[l] \arrow[r] & G_\bullet \end{tikzcd}$$

which leads to a morphism of one-winged butterflies

$$\begin{tikzcd} & \Delta^*\big((E \times E') \times^{G_1 \times G_1}_{G_0 \times G_0} P\big) \arrow[dl] \arrow[dr] \arrow[dd] & \\ \Gamma & & G_\bullet \\ & \Delta^*\big((E' \times E) \times^{G_1 \times G_1}_{G_0 \times G_0} P\big) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

from Γ to $G_\bullet$. This provides a purely diagrammatic proof that the group structure of $\mathrm{H}^1(\mathrm{B}\Gamma, \mathscr{G})$ is abelian when $\mathscr{G}$ is symmetric. At the level of diagrams, it is a braiding on the category $\mathsf{Ext}(\Gamma, G_\bullet)$.

8.3 Butterflies, extensions, and simplicial morphisms

Consider again a generic morphism $F\colon \mathscr{H} \to \mathscr{G}$ of gr-stacks and the corresponding butterfly (2.2.1). Using a sheafified nerve construction, F corresponds to a simplicial map

$$\overline{\mathrm{W}}\,\underline{H}_\bullet \longrightarrow \overline{\mathrm{W}}\,\underline{G}_\bullet, \tag{8.3.1}$$

via the map $\underline{H}_\bullet \to \underline{G}_\bullet$ in the sense of A_∞-spaces, thanks to considerations analogous to those of [Bre90, §8.5]. In the set-theoretic case this simplicial map is the starting point for the definition of weak-morphism of crossed module, which is then *computed* by a butterfly diagram. In the sheaf-theoretic context the starting point for the definition of weak morphism is different (See the discussion in [Part I, §4.2]). Thus, it is of some interest to re-obtain the simplicial map in the present context.

Rather than appealing to A_∞-geometry, we sketch a different way to arrive at the same conclusion, as follows. If in the butterfly (2.2.1) we isolate the "one-winged" one,

$$\begin{tikzcd} & & G_1 \arrow[dl, "\imath"'] \arrow[dd, "\partial"] \\ & E \arrow[dl, "\pi"'] \arrow[dr, "\jmath"] & \\ H_0 & & G_0 \end{tikzcd} \tag{8.3.2}$$

analogous to (8.1.1), we obtain a class in $\mathrm{H}^1(\mathrm{B}H_0, \mathscr{G})$, corresponding to a well-defined morphism

$$H_0 \longrightarrow \mathscr{G},$$

in the sense of gr-stacks. Thus, the underlying geometric object to the extension (8.3.2) is a $\mathscr{G}$-torsor, or equivalently, a gerbe bound by $G_\bullet$, over $\mathrm{B}H_0$.

Next, the standard pull-back (see [ML95]) of the extension (8.3.2) to H_1 via $\partial\colon H_1 \to H_0$ is trivial, due to the existence of the homomorphism $\kappa\colon H_1 \to E$ in the full butterfly (2.2.1). It follows that the class of the extension (8.3.2) dies under the pull-back map

$$(\mathrm{B}\partial)^*\colon\ \mathrm{H}^1(\mathrm{B}H_0, \mathscr{G}) \longrightarrow \mathrm{H}^1(\mathrm{B}H_1, \mathscr{G}). \tag{8.3.3}$$

The condition that the pullback of the cocycle corresponding to the extension (8.3.2) vanish leads to an explicit simplicial map (8.3.1). The actual computation via cocycles is uneventful and quite laborious, so we omit it.

More interesting is the geometric reason, which we record in the following informal assertions—not all verification having being carried out. Essentially, the $\mathscr{G}$-torsor over $\mathrm{B}H_0$ defined by the extension (8.3.2) "descends" to $\overline{\mathrm{W}}\,\underline{H}_\bullet$ along the map $\mathrm{B}H_0 \to \overline{\mathrm{W}}\,\underline{H}_\bullet$.

8.3.4 Assertion. The vanishing of the image of the class of the extension (8.3.2) under the map (8.3.3) determines 2-descent data for the $\mathscr{G}$-torsor determined by the extension (8.3.2) relative to the map $\mathrm{B}H_0 \to \overline{\mathrm{W}}\,\underline{H}_\bullet$.

Sketch of the proof. Consider the augmented (bi)simplicial object

$$U_{\bullet\bullet} = \operatorname{cosk}_0\bigl(\mathrm{B}H_0 \to \overline{\mathrm{W}}\,\underline{H}_\bullet\bigr)\colon\ \cdots \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathrm{B}H_0 \times_{\overline{\mathrm{W}}\,\underline{H}_\bullet} \mathrm{B}H_0 \rightrightarrows \mathrm{B}H_0 \longrightarrow \overline{\mathrm{W}}\,\underline{H}_\bullet$$

where the first index is the "external" one, whose face maps are explicitly drawn above. We compute $BH_0 \times_{\overline{\mathrm{W}}\,\underline{H}_\bullet} BH_0 \simeq \mathrm{B}(H_0 \ltimes H_1)$, and so on, therefore $U_{\bullet\bullet}$ is equivalent to B applied degree-wise to $\underline{H}_\bullet$:

$$\cdots \substack{\longrightarrow\\ \vdots\\ \longrightarrow} \mathrm{B}(H_0 \ltimes (H_0 \ltimes H_1)) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathrm{B}(H_0 \ltimes H_1) \rightrightarrows BH_0 \longrightarrow \overline{\mathrm{W}}\,\underline{H}_\bullet$$

The face maps are actually induced by those of $\underline{H}_\bullet$. Note that the diagonal of the above bisimplicial object is equivalent to $\overline{\mathrm{W}}\underline{H}_\bullet$.

The extension (8.3.2) determines a bitorsor cocycle of the type (8.1.5) which we write as:

$$\gamma_{x,y} \colon E_{xy} \xrightarrow{\sim} E_x \overset{G_1}{\wedge} E_y,$$

for points x, y of H_0. The class of this cocycle is trivial under the pull-back (8.3.3), and moreover we know the pulled-back extension is actually a *direct* product, rather than merely a semi-direct one, since the composition $\jmath \circ \kappa$ is trivial in the full butterfly. A moment's thought reveals the (G_1, G_0)-torsor determined by a direct product extension is in fact trivial, i.e. of the form $(G_1, 1)$, hence we must have coherent isomorphisms

$$\delta_h \colon E_{\partial h} \xrightarrow{\sim} G_1,$$

where of course $E_{\partial h}$ is the "value" of the pulled back cocycle at h.

At a point (y, h) of $H_0 \ltimes H_1$, the pull-backs of E along the two face maps

$$d_i \colon H_0 \ltimes H_1 \longrightarrow H_0, \quad i = 0, 1,$$

$d_0(y, h) = y\partial h$, and $d_1(y, h) = y$, are:

$$d_0^* E_{(y,h)} = E_{y\partial h}, \quad d_1^* E_{(y,h)} = E_y.$$

Using the cocycle condition and the triviality argument above, we have an isomorphism

$$E_{y\partial h} \xrightarrow{\gamma_{y,\partial h}} E_y \overset{G_1}{\wedge} E_{\partial h} \xrightarrow{1 \wedge \delta_h} E_y$$

at each point (y, h) of $H_0 \ltimes H_1$. Thus, we have obtained an isomorphism of extensions, and hence of $\mathscr{G}$-torsors, or again gerbes bound by $G_\bullet$, over the first stage $U_{1\bullet}$.

Similar arguments, this time using the coherence of γ and δ, would show the axioms of a 2-descent datum with respect to $\mathrm{B}H_0 \to \overline{\mathrm{W}}\underline{H}_\bullet$ are satisfied. □

Let us denote by $\mathscr{E}$ the descended gerbe over $\overline{\mathrm{W}}\underline{H}_\bullet$. Finally we have:

8.3.5 Assertion. The class of $\mathscr{E}$ determines the simplicial map (8.3.1).

Sketch of the proof. After sections 3 and 5, the class of a gerbe is effectively a simplicial map of the sought-after type. □

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